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Date

Summary: All Theorems, Lemma and Corollary of Poisson Process

Durrette:

Section 2.1. Exponential Distribution:

Def: $T = \text{exponential}(\lambda)$ if CDF is: $P(T \le t) = 1 - e^{-\lambda t}$ for all $t \ge 0$

PDF is: $f_T(t) = \begin{cases} \lambda e^{-\lambda t}, & t \ge 0 \\ 0, & t < 0 \end{cases}$

$E(T) = 1/\lambda \quad E(T^2) = 2/\lambda^2 \quad Var(T) = E(T^2) - [E(T)]^2 = 1/\lambda^2$

Property: ① Lack of memory: $P(T > t+s \mid T > t) = P(T > s)$

② Exponential races: $S = \text{exponential}(\lambda)$ $T = \text{exponential}(\mu)$

Then $\min(S, T) \sim \text{exponential}(\lambda + \mu)$

And $P(S < T) = \int_0^\infty f_S(s) P(T > s) ds = \frac{\lambda}{\lambda + \mu}$

*: Important integral: $\int_0^\infty t e^{-tx} dx = 1$, t is a constant

Theorem 2.1: $V = \min(T_1, \cdots, T_n)$ and I be the index of smallest one.

$T_i \sim \text{Exponential}(\lambda_i)$, then: $\begin{cases} P(V > t) = \exp(-(\lambda_1 + \cdots + \lambda_n)t) \\ P(I = i) = \lambda_i / (\lambda_1 + \cdots + \lambda_n) \end{cases}$

Section 2.2: Defining the Poisson Process

Def of Poisson Distribution: If $X \sim \text{Poisson}(\lambda)$, then:

$P(X = n) = e^{-\lambda} \frac{\lambda^n}{n!}$ for $n = 0, 1, 2 \cdots$ and $E(X) = Var(X) = \lambda$

Theorem 2.2: For $\forall k \ge 1$, $E(X(X-1) \cdots (X-k+1)) = \lambda^k$

2.3: If $X_i \sim \text{Poisson}(\lambda_i)$, then $\sum_{i=1}^n X_i \sim \text{Poisson}(\sum_{i=1}^n \lambda_i)$

* To prove 2.3, delve into $n=2$, then generalize

Def of Poisson Process (PP): $\{N(s), s \ge 0\}$ is a PP if:

① $N(0) = 0$ ② $N(t+s) - N(t) = \text{Poisson}(\lambda s)$

③ $N(t)$ has independent increments, i.e., if $t_0 < t_1 < \cdots < t_n$, then:

$N(t_1) - N(t_0), \cdots, N(t_n) - N(t_{n-1})$ are independent

Theorem 2.4: If n is large, then binomial $(n, \lambda/n)^* \approx \text{Poisson}(\lambda)$

* $P(X=k) = \binom{n}{k} \left(\frac{\lambda}{n}\right)^k \left(1 - \frac{\lambda}{n}\right)^{n-k} = \frac{n(n-1) \cdots (n-k+1)}{k!} \left(\frac{\lambda}{n}\right)^k \left(1 - \frac{\lambda}{n}\right)^{n-k}$

$\triangle$ $\left(1 - \frac{\lambda}{n}\right)^n \to e^{-\lambda}$ as $n \to \infty$

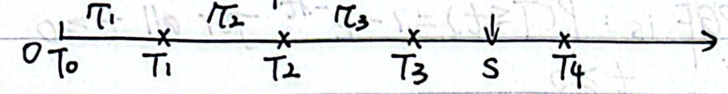
KOKUYO

No.

Date

Section 2.2.1: Constructing the Poisson Process

Def: Let $\tau_1, \tau_2, \cdots$ be i.i.d. exponential(λ) r.v.s. $T_n = \tau_1 + \cdots + \tau_n$, $T_0 = 0$, and define $N(s) = \max\{n : T_n \le s\}$.



Eg. $\underset{\downarrow}{N(s)} = 3$
of τ points before s

Theorem 2.5: Let $\tau_1, \tau_2 \cdots$ be independent exponential(λ). Then:

$T_n = \tau_1 + \cdots + \tau_n \sim$ gamma(n, λ), with PDF as: $\cancel{\&}$ CDF: $F_{T_n}(t) = 1 - \sum_{k=0}^{n-1} \frac{(\lambda t)^k e^{-\lambda t}}{k!}$

$$f_{T_n}(t) = \lambda e^{-\lambda t} \frac{(\lambda t)^{n-1}}{(n-1)!} \quad \text{for } t \ge 0$$

$[N(s) \sim \text{Poisson}(\lambda s)]$

Lemma 2.6: $N(s)$ has a Poisson Distribution with mean λs

2.7: $N(t+s) - N(s)$, $t \ge 0$ is a rate λ PP and [independent of $N(r)$, $r \in [0, s]$]* Markov Property

Section 2.3 Compound Poisson Process

Theorem 2.10: Let $Y_1, Y_2 \cdots$ be independent and identically distributed. Let N be an independent non-negative r.v.. $S = Y_1 + \cdots + Y_N$ with $S \ge 0$ if $N = 0$. Then:

① If $E|Y_i|$, $E(N) < \infty$, then $E(S) = E(N) \cdot E(Y_i)$

② If $E|Y_i|^2$, $E(N^2) < \infty$, then $var(S) = E(N) \cdot var(Y_i) + var(N) \cdot [E(Y_i)]^2$

Now, if $N \sim$ Poisson(λ), then $var(S) = \lambda E(Y_i^2)$

Section 2.4 Transformations

2.4.1: Let $N_j(t)$ be the number of i s.t. $i \le N(t)$ and $Y_i = j$. Then:

Theorem 2.11: $N_j(t)$ are independent rate $\lambda P(Y_i = j)$ PP

2.12: Suppose a PP with λ, we keep the point lands at s with $p(s)$. Then result is a nonhomogeneous PP with rate $\lambda p(s)$*

*: Def: $\{N(s), s \ge 0\}$ with rate $\lambda(r)$ if ① $N(0) = 0$ ② $N(t)$ has independent increment ③ $N(t) - N(s)$ is a poisson distribution with mean $\int_s^t \lambda(r) dr$

and mean $= \mu$ ('end by', sort of like CDF)

Example: Start of Phone call has PP, and duration: P(start at s and end at t) $= G(t-s)$. Then the number of call in progress at time t:

Expectation $= \int_{s=0}^{t} \lambda \cdot \underbrace{(1 - G(t-s))}_{\text{prob of keeping a point}} = \lambda \int_{r=0}^{t} (1 - G(r)) dr$

No.

Date . .

Theorem 2.13. In the phone call with $t \to \infty$:

$$\lambda \int_{r=0}^{\infty} (1 - G(r)) dr = \lambda \mu \ ✡$$

✡: $X \in \mathbb{Z}^+$, then $E(X) = \sum_{k=1}^{\infty} P(X \geq k)$.

2.4.2 Superposition

Going in the other direction and adding up a lot of independent process is called superposition.

Theorem 2.14: Suppose $N_1(t), \cdots, N_k(t)$ are independent PP with rates $\lambda_1, \cdots, \lambda_k$, then $N_1(t) + \cdots + N_k(t)$ is a PP with rate $\lambda_1 + \cdots + \lambda_k$

2.4.3 Conditioning

Let $T_1, T_2, T_3 \cdots$ be the arrival times of a PP with rate λ, let $U_1, U_2, \cdots$ be independent and uniformly distributed on $[0, t]$, let $V_1 < \cdots < V_n$ be the U_i rearranged in increasing order.

Theorem 2.15: If we condition on $N(t) = n$, then:

$(T_1, T_2, \cdots)$ has the same distribution as $(V_1, V_2 \cdots)$

So set $\{T_1, \cdots, T_n\}$ has the same distribution as $\{U_1, \cdots, U_n\}$

Theorem 2.16: If $s < t$ and $0 \leq m \leq n$, then:

$$P(N(s) = m \mid N(t) = n) = \frac{\underbrace{P(N(s) = m)}_{\text{Poisson}(s\lambda)} \ \underbrace{P(N(t) - N(s) = n-m)}_{\text{Poisson}((t-s)\lambda)}}{\underbrace{P(N(t) = n)}_{\text{Poisson}(t\lambda)}}$$

$$= \frac{e^{-\lambda s} (\lambda s)^m / m! \cdot e^{-\lambda(t-s)} [\lambda(t-s)]^{n-m} / (n-m)!}{e^{-\lambda t} (\lambda t)^n / n!}$$

$$= \binom{n}{m} \left(\frac{s}{t}\right)^m \left(\frac{t-s}{t}\right)^{n-m}$$

Pinsky and Karlin: (Supplements).

5.1 Poisson Distribution and Poisson Process

Theorem 5.2: Let N be a poisson r.v. with μ, and conditioned on N, let M have a binomial distribution with N & p. Then the unconditional distribution of M is poisson with parameter μp.

*: During proof: $\sum_{n=0}^{\infty} \frac{t^n}{n!} \to e^t$ taylor expansion

5.2 The Law of Rare events

Claim: $X \sim \text{Binomial}(N, p)$, let $X_{N,p}$ denote # of success in N trials.

$$\Pr\{X_{N,p} = k\} = \binom{N}{k} p^k (1-p)^{N-k}$$

If $N \to \infty$, $p \to 0$ in a way s.t. $Np = \mu > 0$, then $X_{N,p} \approx \text{Poisson}(\lambda)$

$$\Pr\{X_\mu = k\} = \frac{e^{-\mu}\mu^k}{k!}$$

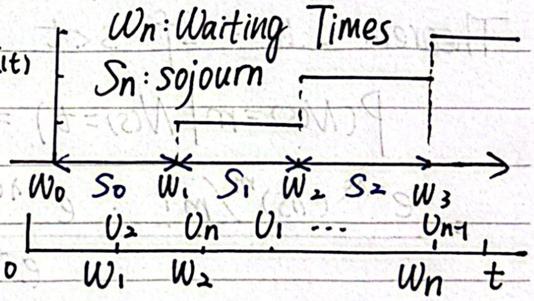
But if every trial ϵ_i has different prob? $\Pr\{\epsilon_i = 1\} = p_i$, then: $S_n = \epsilon_1 + \cdots + \epsilon_n$

$$\Pr\{S_n = k\} = \sum^{(k)} \prod_{i=1}^{n} p_i^{x_i} (1-p_i)^{1-x_i}$$

where $\sum^{(k)}$ denotes the sum over 0 & 1 s.t. $x_1 + \cdots + x_n = k$.

Then: Theorem 5.3: consider poisson process ($\mu = p_1 + \cdots + p_n$) for approximation.

$$\left| \Pr\{S_n = k\} - \frac{\mu^k e^{-\mu}}{k!} \right| \le \sum_{i=1}^{n} p_i^2$$



5.3 Distributions Associated with PP

5.4 The uniform distribution and PP

Consider throwing dart at $[0, t]$ each marked U_i. So $f_U(u) = \frac{1}{t}$ if $u \in [0, t]$

Let $W_1 \le W_2 \le \cdots \le W_n$ denote $\{U_i\}$ in order. So obviously joint PDF of W_i:

$$f_{W_1, \cdots, W_n}(w_1, \cdots, w_n) = n!\, t^{-n}, \quad w_1, w_2, \cdots w_n \in [0, t], \; w_1 \le w_2 \cdots \le w_n$$

Theorem 5.7: Let $W_1, W_2 \cdots$ be the occurance times in PP of rate λ. Conditioned on $N(t) = n$, then:

$$f_{W_1, \cdots, W_n | N(t) = n}(w_1, \cdots, w_n) = n!\, t^{-n} \quad \text{for } 0 < w_1 < \cdots < w_n \le t$$

Has important applications in evaluating certain symmetric functionals on PP.