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STAT 150 Poisson Process.

$\boxed{5.1}$ Exponential Distribution : $P(T \le t) = 1 - e^{-\lambda t}$, $t \ge 0$

(Durette) PDF: $f_T(t) = \begin{cases} \lambda e^{-\lambda t}, & t \ge 0 \\ 0, & \text{else} \end{cases}$

$\therefore E(T) = 1/\lambda \quad E(T^2) = 2/\lambda^2, \quad Var(T) = 1/\lambda^2$

Properties: ① Lack of memory: $T \sim$ Exp Dist with λ. then

$P(T > t+s \mid T > t) = P(T > s)$

② Exponential races: $S = exp(\lambda)$ $T = exp(\mu)$, then:

$P(\min(S,T) > t) = e^{-(\lambda+\mu)t}$, i.e., $\min(S,T) \sim Exp(\lambda+\mu)$

And $P(S<T) = \int_0^\infty f_S(s) P(T>s) ds = \frac{\lambda}{\lambda+\mu}$

Theorem: Let $V = \min(T_1, \cdots, T_n)$ and I be the (random) index of the T_i that is smallest.

$P(V>t) = exp(-(\lambda_1 + \cdots + \lambda_n)t)$

$P(I=i) = \lambda_i / (\lambda_1 + \cdots + \lambda_n)$

With Exp Dist, here to define Poisson Process

Def : We say X has a poisson distribution with mean λ, or $X = poisson(\lambda)$ if: $P(X=n) = e^{-\lambda} \frac{\lambda^n}{n!}$ for $n = 0, 1, \cdots$

Theorem (of Poisson): For any $k \ge 1$: $EX(X-1)\cdots(X-k+1) = \lambda^k$, and hence $var(X) = \lambda$

Theorem: If X_i are independent Poisson(λ_i) then

$X_1 + \cdots + X_k = Poisson(\lambda_1 + \cdots + \lambda_n)$

Def: $\{N(s), s \ge 0\}$ is a Poisson process, if:

(i) $N(0) = 0$ (ii) $N(t+s) - N(s) = Poisson(\lambda t)$

(iii) $N(t)$ has independent increment, i.e., $t_0 < t_1 < \cdots < t_n$. then: $N(t_1) - N(t_0), \cdots N(t_n) - N(t_{n-1})$ are independent

Theorem: If n is large, then binomial$(n, \lambda/n)$ dist is approximately poisson(λ)

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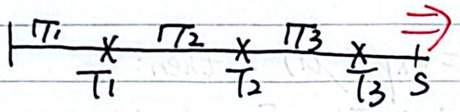
Constucting a PP:

Def: Let $\tau_1, \tau_2 \cdots$ be indep $\exp(\lambda)$ dist r.v.s, let:
$T_n = \tau_1 + \cdots + \tau_n$ for $n \geq 1$, $T_0 = 0$, then define:

$$N(s) = \max\{n : T_n \leq s\}.$$

Think of τ_n as times between arrival. $T_n = \tau_1 + \cdots + \tau_n$ is the arrival time of the n-th customer. $N(s)$ is the number of arrivals by times.



$\Rightarrow N(s) = 3.$, $T_3 \leq s < T_4$.

Theorem: Let $\tau_1, \tau_2, \cdots \tau_n$ be indep $Exp(\lambda)$. Then $T_n = \sum_{i=1}^{n} \tau_i$ has a gamma (n, λ) distribution. That is:

$$f_{T_n}(t) = \lambda e^{-\lambda t} \frac{(\lambda t)^{n-1}}{(n-1)!} \quad \text{for } t \geq 0$$

Lemma: $N(s)$ has a Poisson Distribution with mean λs. $(0 \leq r < s)$
$N(t+s) - N(s)$ is a rate λ Poisson Process and independent of $N(r)$.
$\Rightarrow N(t)$ has independent increments.

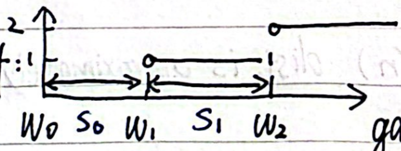
Nonhomogeneous Poisson Process: We say $\{N(s), s \geq 0\}$ is a PP with rate $\lambda(r)$ if (i) $N(0) = 0$ (ii) $N(t)$ has independent increment and (iii) $N(t) - N(s)$ is Poisson with mean $\int_s^t \lambda(r) dr$

$\boxed{5.2}$ Theorem: Let $\epsilon_1\ \epsilon_2 \cdots$ be independent Bernoulli random variables, where $Pr\{\epsilon_i = 1\} = p_i$ $Pr\{\epsilon_i = 0\} = 1 - p_i$.

Let $S_n = \epsilon_1 + \cdots + \epsilon_n$. The exact probabilities for S_n with $\mu_n = p_1 + \cdots + p_n$ differ by at most:

$$\left| Pr\{S_n = k\} - \frac{\mu^k e^{-\mu}}{k!} \right| \leq \sum_{i=1}^{n} p_i^2$$

$\boxed{5.3}$ Def:



Theorem: The waiting time W_n has gamma distribution whose PDF is:

$$f_{W_n}(t) = \frac{\lambda^n t^{n-1}}{(n-1)!} e^{-\lambda t}, \quad n = 1, 2 \cdots t \geq 0$$

In particular: $f_{W_1}(t) = \lambda e^{-\lambda t} \sim$ Exponential (λ)

Very useful PDF! Pf: $F_{W_n}(t) = Pr\{W_n \le t\} = Pr\{X(t) \ge n\}$

And $Pr\{X(t)=k\} \sim$ Poisson (λ)

$$\therefore P(X(t)=k) = \frac{(\lambda t)^k e^{-\lambda t}}{k!} \qquad = \sum_{k=n}^{\infty} \frac{(\lambda t)^k e^{-\lambda t}}{k!} = 1 - \sum_{k=0}^{n-1} \frac{(\lambda t)^k e^{-\lambda t}}{k!}$$

And we know: $f_{W_n}(t) = \dfrac{dF_{W_n}(t)}{dt}$

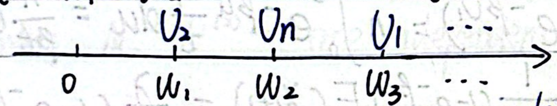
Meanwhile: S_n are called sojourn times, and $f_{S_k}(s) = \lambda e^{-\lambda s}, s \ge 0$

Also: $Pr\{X(u)=k \mid X(t)=n\}_{(0<u<t)} = \dfrac{n!}{k!(n-k)!}\left(\dfrac{u}{t}\right)^k\left(1-\dfrac{u}{t}\right)^{n-k}$ ~Binomial $(n, u/t)$

Proof: $= \dfrac{Pr\{X(t)-X(u)=n-k,\ X(u)=k\}}{Pr\{X(t)=n\}} = \dfrac{(e^{-\lambda u}(\lambda u)^k/k!)(e^{-\lambda(t-u)}[\lambda(t-u)]^{n-k}/(n-k)!)}{e^{-\lambda t}(\lambda t)^n/n!}$

$\boxed{5.4}$ Uniform Distribution and Poisson Process

Let $W_1 \le W_2 \cdots \le W_n \le t$. Let U_i be the i-th dart thrown, and $f_U(u) = \begin{cases} 1/t \\ 0 \end{cases}$



The the joint PDF of $W_1, \cdots, W_n$ is:

$$f_{W_1,\cdots,W_n}(w_1,\cdots,w_n) = n!\,t^{-n}$$

Also theorem: Let $W_1, \cdots, W_n$ be the occurence times in PP of rate $\lambda > 0$. Conditioned on $N(t)=n$:

Theo: equivalent

$$f_{W_1,\cdots,W_n \mid X(t)=n}(w_1,\cdots,w_n) = n!\,t^{-n} \quad \text{for } 0<w_1<\cdots<w_n \le t$$

$\boxed{5.4.1}$ Assume: ① Electrons arrive at an anode according to PP with λ.

② An arriving electron produces a current whose intensity x time units after arrival is given by $h(x)$

$\therefore I(t) = \sum_{k=1}^{X(t)} h(t-W_k)$ Suppose $h(x) = e^{-\theta x}$, and there are noise

$h(x) = x^{-\theta}$

We use uniform distribution over interval $(0,t]$ and def $\epsilon_k = h(U_k)$

$\therefore S(t) = \epsilon_1 + \cdots + \epsilon_{X(t)} = I(t)$

$E[I(t)] = E[S(t)] = \lambda t E[h(U_1)] = \lambda \int_0^t h(u)du$

Similarly $Var(I(t)) = \cdots = \lambda \int_0^t h(u)^2 du$

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Also $\boxed{Pr\{I(t) \le x\}} = Pr\{\sum_{k=1}^{X(t)} h(t - X(k)) \le x\}$

$= \sum_{n=0}^{\infty} Pr\{\sum_{k=1}^{n} h(t-W_k) \le x \mid X(t)=n\} Pr\{X(t)=n\}$.

With Theorem in 5.4 and essentiality equivalance:

$= \sum_{n=0}^{\infty} Pr\{\sum_{k=1}^{n} h(t-U_k) \le x\} Pr\{X(t)=n\}$.

$= \sum_{n=0}^{\infty} Pr\{\sum_{k=1}^{n} h(U_k) \le x\} Pr\{X(t)=n\} = Pr\{\epsilon_1 + \cdots + \epsilon_{X(t)} \le x\}$ $\boxed{= Pr\{S(t) \le x\}}$.

Exercise: Customers arrive according to PP of rate λ. Each pay 1\$ at arrival and it is desired to evaluate the expected value of total sum collected during $(0,t]$ discounted back to time t. This quantity is given by:

$$M = E\left[\sum_{k=1}^{X(t)} e^{-\beta W_k}\right]$$

where $W_1, \cdots$ are arrival times. Then:

$$M = \sum_{n=1}^{\infty} \underbrace{E\left(\sum_{k=1}^{n} e^{-\beta W_k}\right)} \cdot P(X(t)=n)$$

$\hookrightarrow = E\left(\sum_{k=1}^{n} e^{-\beta U_k}\right) = n \cdot E(e^{-\beta U_1}) = n \int_0^t e^{-\beta u} \frac{1}{t} du = \frac{n}{\beta t}(1-e^{-\beta t})$.

$\therefore M = \frac{n}{\beta t}(1-e^{-\beta t}) \sum_{n=1}^{\infty} P(X(t)=n) = \frac{1}{\beta t}(1-e^{-\beta t}) E(X(t)) = \frac{\lambda}{\beta}(1-e^{-\beta t})$

Theorem: Superposition Theorem: $\pi_i = \{\tau_1^{(i)} < \tau_2^{(i)} < \cdots\}$, $\pi_i \sim PP(\lambda_i)$ s.t. $\sum_{i=1}^{\infty} \lambda_i < \infty$. Then: $\bigcup_{i=1}^{\infty} \pi_i \sim PP(\sum_{i=1}^{\infty} \lambda_i)$

$\boxed{5.5}$ Spatial Poisson Processes

Let S be a subset of real line, two-dim plane or three-dim space; let $\mathcal{A}$ be the family of subsets in S, and for any A in $\mathcal{A}$, let $|A|$ denote the size of A. Then $\{N(A); A \text{ in } \mathcal{A}\}$ is a homogeneous point process of intensity $\lambda > 0$ if:

1) for each A in $\mathcal{A}$, r.v. $N(A)$ has a Poisson distribution with parameter $\lambda|A|$

2) for every finite collection $\{A_1, \cdots, A_n\}$ of disjoint subsets of S, the r.v. $N(A_1), \cdots, N(A_n)$ are independent.

So: $Pr\{N(A)=k\} = \dfrac{e^{-\lambda|A|}(\lambda|A|)^k}{k!}$ and $Pr\{N(B)=1 \mid N(A)=1\} = \dfrac{|B|}{|A|}$ for $\forall B \subseteq A$

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5.6 Compound and Marked Poisson Process

Given PP $X(t)$ with λ. suppose each event has associated with it with a r.v. representing something. The successive values $Y_1, Y_2 ..$ are assumed to be independent and r.v. sharing the common distribution function:

$$G(y) = Pr\{Y_k \le y\}$$

And a compound PP process is the cumulative value process defined by:

$$Z(t) = \sum_{k=1}^{X(t)} Y_k$$

A marked PP is the sequence of pairs (W_i, Y_i) ...

5.6.1 Compound PP: $Z(t) = \sum_{k=1}^{X(t)} Y_k$. If $\mu = E(Y_1)$, $\nu^2 = Var(Y_1)$, then:

$E[Z(t)] = \lambda \mu t$, $Var[Z(t)] = \lambda t(\nu^2 + \mu^2)$

5.6.2 Marked PP: $G(y) = Pr\{Y_k \le y\}$. Mark each point of N by $1, 2, 3 \cdots$

with prob: $p_1, p_2, \cdots$. $p_i > 0$ and $\sum_{i=1}^{\infty} p_i = \infty$

Let $N^{(i)} = (N_t^{(i)})$ s.t. $N_t^{(i)} = \#$ of points in $[0, t]$ marked with i

Then $N^{(1)} N^{(2)} \cdots$ are independent PP with λp_i rate.