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Martingales

5.1 Conditional Expectation

Def: Given an event A we define its indicator function:

$$\mathbb{1}_A = \begin{cases} 1, & x \in A \\ 0, & x \in A^c \end{cases}$$

Given a r.v. Y, we define the integral of Y over A:

$E(Y; A) = E(Y \mathbb{1}_A)$ "multiplying"

Finally: $E(Y|A) = \dfrac{E(Y;A)}{P(A)} \Leftrightarrow P(\cdot|A) = \dfrac{P(\cdot \cap A)}{P(A)}$

Lemma 5.1: If X is a constant c on A, then $E(XY/A) = cE(Y|A)$

Lemma 5.2: Jensen's Inequality: If $\phi(\cdot)$ is convex, then:

$E(\phi(X)|A) \geq \phi(E(X|A))$

Lemma 5.3: If B is a disjoint union of $A_1, \cdots, A_k$, then:

$E(Y;B) = \sum_{i=1}^{k} E(Y; A_i)$

Lemma 5.4: If B is a disjoint union of $A_1, \cdots, A_k$, then:

$E(Y|B) = \sum_{i=1}^{k} E(Y|A_i) \cdot P(A_i)$

5.2. Theorem 5.5: Let X_n be a Markov chain with transition probability p and let $f(x,n)$ be a function of the state x and time n, s.t.: $f(x,n) \geq \sum_y p(x,y) f(y, n+1)$

Then: $M_n = f(X_n, n)$ is a supermartingale w.r.t. X_n.

Wait: Def: Regarding M_n as the amount of money at time n for a gambler betting on a fair game, X_n as the outcomes of the game, we say that $M_0, M_1, \cdots$ is a martingale w.r.t. $X_0, X_1, \cdots$ if (i). $\forall n \geq 0$, $E|M_n| < \infty$ (ii). Value of M_n can be determined from values for $X_n, \cdots, X_0$ and M_0 (iii) For any possible values $x_n, \cdots, x_0$:

$E(M_{n+1} - M_n \mid X_n = x_n, X_{n-1} = x_{n-1}, \cdots, X_0 = x_0, M_0 = m_0) = 0$ is 0.

$\Downarrow$ Conditioned on the past up to time n, the average profit on nth game

Sometimes we use: $A_v = \{X_n = x_n, \cdots, X_0 = x_0, M_0 = m_0\}$, $v: (x_n, \cdots, x_0, m_0)$

Random walks:

Def: Let $X_1, X_2, \cdots$ be i.i.d with $EX_i = \mu$. Let $S_n = S_0 + X_1 + \cdots + X_n$ be a random walk. Then $M_n = S_n - n\mu$ is a martingles w.r.t X_n.

$\Rightarrow$ Linear Martingale

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Def:

But if $E(M_{n+1} - M_n | A_v) \le 0 \Rightarrow$ supermartingale

if $E(M_{n+1} - M_n | A_v) \ge 0 \Rightarrow$ submartingale

Def: $X_n \cdots$ are i.i.d with $E(X_i) = 0$, $var(X_i) = \sigma^2$. Then: $S_n = \sum_{i=0}^{n} X_i + S_0$; $M_n = S_n^2 - n\sigma^2$ is also a martingale w.r.t. X_n. $\Rightarrow$ Quadratic Martingale

Def: $Y_1, \cdots$ are i.i.d with $\phi(\theta) = E[\exp(\theta Y_i)] < \infty$. Let $S_n = S_0 + \sum_{i=1}^{n} Y_i$.

Then $M_n = \exp(\theta S_n) / \phi(\theta)^n$ is a martingale w.r.t. Y_n. $\Rightarrow$ Exponential Martingale

Theorem 5.6/7. If M_m is a supermartingale and $m \le n$, then $EM_m \ge EM_n$

If M_n sub, then $EM_m \le EM_n$

Theorem 5.8. If M_m is a martingale and $0 \le m < n$, then $EM_m = EM_n$

5.3. Gambling Strategies, Stopping Times

Theorem 5.9. Suppose that M_n is a supermartingale w.r.t. X_n, H_n is predictable, and $0 \le H_n \le c_n$, where c_n is a constant depending on n.

Then: $W_n = W_0 + \sum_{m=1}^{n} H_m (M_m - M_{m-1})$ is a supermartingale.

One possible strategy is to have a stopping time T. $H_m = \begin{cases} 1, & \text{if } T \ge m \\ 0, & \text{else.} \end{cases}$

Then: Theorem 5.10: If M_n is a supermartingale w.r.t. X_n and T is a stopping time, then $M_{T \wedge n}$ is a supermatingale w.r.t. X_n:

i.e. $EM_{T\wedge n} \le EM_0$ $\quad T\wedge n \rightarrow$ denote $\min\{T, n\}$

5.4.1 Application

Theorem 5.11: M_n is a martingale and T is a stopping time (finite), and $|M_{T\wedge n}| \le K$ for some constant K. Then $EM_T = EM_0$

Let X_n be a Markov Chain with State Space S and $A, B \subset S$. Let $V_A = \min\{n \ge 0 : X_n \in A\}$ and suppose that $P_x(V_A \wedge V_B < \infty) > 0$ for all $x \in C$

Then: Theorem 5.12: If $h(a) = 1$ for $a \in A$, $h(b)$ for $b \in A$, and for $x \in C$ we have: $h(x) = \sum_y p(x,y) h(y)$.

then: $h(x) = P_x(V_A < V_B)$. Exit Distribution