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Continuous Markov Chain (CMC)   4.1 Definitions & Examples

Def of CMC:

In discrete time, we formulate markov property as:

$$P(X_{n+1}=j \mid X_n=i, X_{n-1}=x_{n-1}, \cdots, X_0=x_0) = P(X_{n+1}=j \mid X_n=i)$$

But in continuous time, defining all X_r for $r \le s$ can be challenging. So instead, we say that X_t, $t \ge 0$ is a Markov Chain if for any $0 \le s_0 < s_1 \cdots < s_n < s$ and possible states $i_0, \cdots, i_n, i, j$, we have

$$P(X_{t+s}=j \mid X_s=i, X_{s_n}=i_n, \cdots, X_{s_0}=i_0) = P(X_t=j \mid X_0=i)$$

In a word, given the present state, the rest of the past is irrelavant for predicting the future

i.e., prob of going from i at time s to j at time $s+t$, only depends on t

Example. Let $N(t)$, $t \ge 0$ be PP with rate λ and let Y_n be a discrete MC with transition probability $u(i,j)$. Then $X_t = Y_{N(t)}$ is a CMC. In words, X_t takes one jump according to $u(i,j)$ at each arrival time of $N(t)$. X_t transit when pp has a new point at a time t. This is a valid CMC, since in PP point's arrival time is independent, and X_t transite $u(i,j)$ are independent, and independent of time t.

We therefore define: $P_t(i,j) = P(X_t=j \mid X_0=i)$

In the PP example above $P_t(i,j) = \sum_{n=0}^{\infty} \underbrace{e^{-\lambda t}\frac{(\lambda t)^n}{n!}}_{\text{In time span } t\text{, the prob that } X \text{ transit } n \text{ times}} \underbrace{u^n(i,j)}_{\text{prob that } i \to \cdots \to j \text{ with path of length } n}$

Theorem: Chapman - Kolmogorov Equation

$$\sum_k p_s(i,k)\, p_t(k,j) = p_{s+t}(i,j)$$

Intuitively: Yes!

Pf: $P(X_{s+t}=j \mid X_0=i) = \sum_k (X_{s+t}=j, X_s=k \mid X_0=i)$

$= \sum_k P(X_{s+t}=j \mid X_s=k, X_0=i)\, P(X_s=k \mid X_0=i) = \sum_k P_t(k,j)\, P_s(i,k)$

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From Chapman - Kolmogorov, we can know that: if we know the transition probability within $[0, t_0]$, then we will know the transition probability for any t. Because we can iteratively apply this equation.

We will soon justify: $$q(i,j) = \lim_{h \to 0} \frac{p_h(i,j)}{h} \quad \text{for } j \neq i$$

If this limit exists, we will call $q(i,j)$ the jump rate from i to j.

Back to the example of PP. Consider the prob of at least 2 jumps at time h:

$$P = 1 - (e^{-\lambda h} + \lambda h e^{-\lambda h}) = 1 - (1+\lambda h)\left(1 - \lambda h + \frac{(\lambda h)^2}{2!} + \cdots\right)$$

$$= \frac{(\lambda h)^2}{2!} + \cdots = o(h)$$, so if divided by h and $h \to 0$, they $\to 0$

$$\therefore q(i,j) = \lim_{h \to 0} \frac{P_t(i,j)}{h} = \lim_{h \to 0} \frac{e^{-\lambda h}}{h} \cdot u^0(i,j) + \frac{\lambda h e^{-\lambda h}}{h} \cdot u^1(i,j)$$

$$= \lambda u(i,j)$$

Now we may ask: Given the rates, how do we construct chain?

Let $\lambda_i = \sum_{j \neq i} q(i,j)$ be the rate at which X_t leave i. If $\lambda_i = \infty$, then the procedure will want to leave i immediately. So we always suppose each state i has $\lambda_i < \infty$. If $\lambda_i = 0$, then X_t will never leave i. When $\lambda_i > 0$, we get:

$$r(i,j) = q(i,j)/\lambda_i \quad \text{for } j \neq i, \quad \lambda_i = \sum_{j \neq i} q(i,j)$$

is the probability that chain goes to j when it leaves i.

Formal Construction: Suppose $\lambda_i > 0$ for all i. If there are absorbing states, then when chain hits one, it stays forever. Let Y_n be a Markov chain with transition probability $r(i,j)$. The discrete-time chain Y_n, gives the road map that the continuous-time process will follow. To determine how long will the process should stay in each state let $\tau_1, \tau_2, \cdots$ be independent exponentials with rate 1. and:

At time 0: state Y_0 stays for $t_1 = \tau_0 / \lambda(Y_0)$ time

At time t_1: jump to Y_1 and stay for $t_2 = \tau_1 / \lambda(Y_1)$ time ---

$$\therefore \begin{cases} \text{jump to } Y_{n+1} \text{ at time: } T_{n+1} = t_1 + \cdots + t_{n+1} \\ \text{If } T_0 = 0, \text{ then for } n \geq 0, \; X(t) = Y_n \text{ for } T_n \leq t < T_{n+1} \end{cases}$$

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Wrap up: λ_i: the rate of leaving $i \Rightarrow$ time $\propto \frac{1}{\lambda_i}$

τ_i: determine the time of staying

Y_i: record state transition

4.2 Computing the transition probability

$p_t(i,j) = P(X_t = j \mid X_0 = i)$. How to compute prob p_t from the jump rates q_h?

$$p_{t+h}(i,j) - p_t(i,j) = \left[\sum_k p_h(i,k)\,p_t(k,j)\right] - p_t(i,j)$$

$$= \left[\sum_{k\neq i} p_h(i,k)\,p_t(k,j)\right] + \left[p_h(i,i)-1\right] p_t(i,j)$$

Goal: Divide each side by h and $h \to 0$ to compute

$$p_t'(i,j) = \lim_{h\to 0} \frac{p_{t+h}(i,j) - p_t(i,j)}{h}$$

By the def of the jump rates: $q(i,j) = \lim_{h\to 0} \frac{p_h(i,j)}{h}$ for $i \neq j$

So: $\lim_{h\to 0} \frac{1}{h} \sum_{k\neq i} p_h(i,k)\,p_t(k,j) = \sum_{k\neq i} q(i,k)\,p_t(k,j)$

Also: $1 - p_h(i,i) = \sum_{k\neq i} p_h(i,k)$, so:

$$\lim_{h\to 0} \frac{p_h(i,i)-1}{h} = -\lim_{h\to 0} \sum_{k\neq i} \frac{p_h(i,k)}{h} = -\sum_{k\neq i} q(i,k) = -\lambda_i$$

$$\therefore\ p_t'(i,j) = \sum_{k\neq i} q(i,k)\,p_t(k,j) - \lambda_i\,p_t(i,j)$$

To neaten up, we introduce: $Q(i,j) = \begin{cases} q(i,j) & \text{if } j \neq i \\ -\lambda_i & \text{if } j = i \end{cases}$

So $p_t' = Q p_t$

This is called Kolmogorov's backward equation. Inspired:

$$e^{Qt} = \sum_{n=0}^{\infty} Q^n \cdot \frac{t^n}{n!}, \text{ and } \frac{d}{dt} e^{Qt} = Q e^{Qt}$$

Now, revisit: $p_{t+h}(i,j) - p_t(i,j) = \left[\sum_k p_t(i,k)\,p_h(k,j)\right] - p_t(i,j)$

$$= \left[\sum_{k\neq j} p_t(i,k)\,p_h(k,j)\right] + \left[p_h(j,j)-1\right] p_t(i,j)$$

$\cdots$ We have $p_t' = p_t Q = Q p_t$, this is called Kol Forward Equation!

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4.3 Limiting Behavior

Markov chain X_t is irreducible if for any two states i and j, it is possible to get from i to j in finite number of jumps.

Lemma: If X_t is irreducible and $t>0$, then $p_t(i,j)>0$ for all i,j

In CMT, π is said to be a stationary distribution if $\pi p_t = \pi$ for all $t>0$. But p_t can be hard to compute. So:

$$Q(i,j) = \begin{cases} q(i,j), & j \neq i \\ -\lambda_i, & j = i. \end{cases}$$

where $\lambda_i = \sum_{j\neq i} q(i,j)$ is the total rate of transitions out of i.

Lemma: π is a stationary distribution iff $\pi Q = 0$

Theorem: If a CMT X_t is irreducible and has a π, then:

$$\lim_{t\to\infty} p_t(i,j) = \pi(j).$$

Detailed balance condition: We can formulate:

$$\pi(k)\, q(k,j) = \pi(j)\, q(j,k). \quad \text{for all } j \neq k$$

And: Theorem: If $\pi(k)\, q(k,j) = \pi(j)\, q(j,k)$ holds, then π is a stationary distribution.

Important topic: Birth and Death Process

Suppose that $S = \{0, 1, \cdots, N\}$ with $N \leq \infty$, and : $\begin{cases} q(n, n+1) = \lambda_n, & N > n \\ q(n, n-1) = \mu_n, & n > 0 \end{cases}$

So here: λ_n represents the birthrate when there are n individuals in the system, and μ_n denotes the death rate. Formulation

If the birth and death chain is irreducible, and we can divide to write the detailed balance condition as:

$$\pi(n) = \frac{\lambda_{n-1}}{\mu_n} \pi(n-1)$$

i.e.,

$$\pi(n) = \frac{\lambda_{n-1} \cdot \lambda_{n-2} \cdots \lambda_0}{\mu_n \cdot \mu_{n-1} \cdots \mu_1} \pi(0)$$