T1. If $\rho_{xy} = P_x(T_y < \infty) > 0$, but $\rho_{yx} < 1$, x is transient

T2. If C is a finite closed and irreducible set, all states in C are recurrent

T3. If S is finite, S can be $T \cup R_1 \cup R_2 \cdots$ where T is set of trans and R_i are closed irre sets of recu

T4. y is recu iff $\sum_{n=1}^{\infty} P_{yy}^{(n)} = E_y N(y) = \infty$

$N(y)$ be the # of visits to y at times $n \geq 1$

T5. $\lim_{n\to\infty} P_{yy}^{(n)} = 1 / \sum_{n=0}^{\infty} n \rho_{yy}^{(n)}$

T6. If p is doubly stochastic ($\sum_k P_{kj} = 1$), then $\pi(x) = 1/N$ is the unique stationary distribution

T7. I: p is irreducible; A: aperiodic; R: all states recu S: there's a stat dist:

Then: If I, A, S, then $n \to \infty$, $P_{xy}^{(n)} \to \pi(y)$

T8. If I&R, $\frac{N_n(y)}{n} \to \frac{1}{E_y T_y}$

T9. If I&S, $\pi(y) = \frac{1}{E_y T_y}$

T10. I&A&finite $\Leftrightarrow$ Regular, $\exists k_0, \forall k > k_0, p^k$ all > 0

T11. Suppose I, S, and $\sum_x |f(x)| \pi(x) < \infty$, then $\frac{1}{n} \sum_{m=1}^{n} f(X_m) \to \sum_x f(x) \pi(x)$

T12. Suppose I&S: $\frac{1}{n} \sum_{m=1}^{n} P_{xy}^{(m)} \to \pi(y)$

T13. Suppose I & R, $n \to \infty$: $\frac{N_n(y)}{n} \to \frac{1}{E_y T_y}$

T14. $\mu_x(y)$: 在x回x前平均访问y次数:

$\mu_x(y) = \sum_{n=0}^{\infty} P_x(X_n = y, T_x > n)$ 是 stationary measure

i.e. $\mu = (\mu_x(s_1), \mu_x(s_2), \cdots)$ 与π呈倍数关系

T15. Suppose P is I, then: some states are recurrent $\Leftrightarrow$ all are! $\Leftrightarrow$ has a stat dist π

T16. If $0 < p_i < 1$, $i = 0, 1 \cdots$, then $u_m = \prod_{i=0}^{m} (1 - p_i) \to 0$ iff $\sum_{i=0}^{\infty} p_i = \infty$

Notation: $P_{ij}^{(n)}$, ij entry of P^n

or: $P_{ij}^{(n)} = \sum_{k \in S} P_{ik} P_{kj}^{(n-1)}$

$T_{ij} = \min\{n \geq 1 : X_n = j \mid X_0 = i\}$ First visit time to j given $X_0 = i$. If $X_0 = j$, then: $T_{jj} \to T_j$

$T_j^{(k)} = \min\{n > T_j^{(k-1)} : X_n = j\}$. k-th visit time to j

$\rho_{ij} = \mathbb{P}_i(T_j < \infty) = \mathbb{P}(T_j < \infty \mid X_0 = i)$

$\rho_{ii}^{(n)} = \mathbb{P}(T_i = n)$ First visit to j exactly at time n

L1. Suppose $P_x(T_y \leq k) \geq \alpha > 0$ for $\forall x$. then:

$P_x(T_y > nk) \leq (1 - \alpha)^n$

L2: If $x \to y$, $y \to z$, then $x \to z$

L3: If x is recurrent and $\rho_{xy} > 0$, then $\rho_{yx} = 1$

L4: If x is recurrent and $x \to y$, then y is recu

L5: In a finite closed set there has to be at least one recu

L6: $E_x N(y) = \rho_{xy} / (1 - \rho_{yy})$

L7: $E_x N(y) = \sum_{n=1}^{\infty} P_{xy}^{(n)}$

L8: If $\rho_{xy} > 0$ and $\rho_{yx} > 0$, then x y have same period

L9: If $P_{xx} > 0$, then x has period 1

L10: If there's a stat dist, then all states y that have $\pi(y) > 0$ are recurrent

L11: The extinction probability ρ is the smallest solution of $\phi(x) = x$ with $x \in [0, 1]$. $\phi(x)$是1个个体孩子数量的PGF

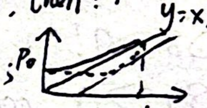
L12: Suppose $\rho = \phi(\rho)$, $\rho \in [0, 1]$, then if $p_1 < 1$:

$\lim_{t\to\infty} X_t = \begin{cases} \infty, & p = 1 - \rho \\ 0, & p = \rho \end{cases}$

$\Pr\{\lim_{n\to\infty} X_n = 0\} = 1$

L13: Suppose $p_1 < 1$, $\mathbb{E}(s) = \mu \leq 1$, $X_0 = 1$, then: ↑

$(0 \leq \phi'(x) \overset{*}{\leq} \phi'(1) = \mu \leq 1$, $*: \phi''(x) > 2p_2 \geq 0$;



L14: Finite irreducible $\Rightarrow$ recurrent! 但 infinite 不一定

Finite irreducible $\Rightarrow$ unique stat dist exist

Infinite irre $\Rightarrow$ $\begin{cases} \text{positive recu: } \checkmark \\ \text{null recu / tran: no!} \end{cases}$

Aperiod or not $\Leftrightarrow$ limiting dist = unique stat dist or doesn't exist

D1: x is Positive Recu if $E_x T_x < \infty$

D2: $\rho_{yy} < 1$, y is transient; $\rho_{yy} = 1$, recu

D3: x communicate with y, $x \to y$, if $\rho_{xy} = P_x(T_y < \infty) > 0$

D4: A is closed if $\forall i \in A$, $j \notin A$, $P_{ij} = 0$; B is irreducible if $\forall i, j \in B$, $i \to j$

D5: Regular: $\exists k_0, \forall k > k_0$, $P_{ij}^{(k)} > 0$ for $\forall i, j \in S$

D6: Stat dist: $\begin{cases} \sum_j \pi_j = 1 \\ \pi = \pi P \end{cases}$

D7: Limiting Dist: $\lim_{n\to\infty} P_{ij}^{(n)} = \pi_j$, or: $\lim_{n\to\infty} \Pr\{X_n = j \mid X_0 = i\} = \pi_j > 0$

$P_{ii}^{(n)} = \sum_{k=0}^{n} \rho_{ii}^{(k)} P_{ii}^{(n-k)}$, Δ: $\rho_{ii}^{(0)} = 0$, $P_{ii}^{(0)} = 1$

$R_j = N_j$: # of visits to j after time ≥ 1; $R_j(n) = N_j(n)$: $\cdots$ time $1 \leq t \leq n$

$r_j(i)$: # of visits to j for time $1 \leq t \leq T_i$ $(X_0 = i)$

$r_j(i) = R_j(T_i)$; $\rho_j(i) = \mathbb{E}(r_j(i)) = \mu_x(y)$ in T14

S1. $\rho_{xy}>0 \Leftrightarrow x\to y \Leftrightarrow \exists k,\ P^{(k)}_{xy}>0$

S2: 证recu: $\begin{cases} P_y(T_y<\infty)=1 \\ \sum_{n=1}^{\infty} P^{(n)}_{xy}=\infty \end{cases}$

S3: Prove x recu, $x\to y \Rightarrow y$ is recu.
If $y\not\to x$, then $x\to y\not\to x$, x tran $\Leftrightarrow y\not\to x$

$\exists k_1,k_2$, $P^{k_1}_{xy}>0$ $P^{k_2}_{xy}>0$, for n:

$\underbrace{\sum_n P^{k_1+n+k_2}_{xx}}_{\infty} \geq \sum_n P^{k_1}_{xy} P^{n}_{yy} P^{k_2}_{yx}$; samely (两视角同理)

$\sum_n P^{k_1+n+k_2}_{yy} \geq \sum_n P^{k_2}_{yx} \underbrace{P^{n}_{xx}}_{\infty} P^{k_1}_{xy} \Rightarrow \sum_n P^n_{yy}=\infty$ y is recu

S4: EX, $X\in\mathbb{Z}^+$, 则 $EX=\sum_{k=1}^{\infty} P(X\geq k)$

S5: $G_X(s)=\mathbb{E}(s^X)=\sum P(X=i)s^i$, then:

$p_0=G_X(0)$; $p_n=\frac{1}{n!}\frac{d^n G_X(s)}{ds^n}\Big|_{s=1}$;

$E(X)=\frac{dG_X(s)}{d(s)}\Big|_{s=1}$; $Var(X)=G''_X(1^-)+G'_X(1^-)-[G'_X(1^-)]^2$

If $X=X_1+X_2\cdots$, $G_X(s)=G_{X_1}(s)G_{X_2}(s)\cdots$

S6. First Step Analysis

S7. If $\gcd(x,y)=1$, $\exists n_0, \forall n>n_0$, $n_0=ax+by$. $a,b\in\mathbb{Z}$

S8. $n!\sim\sqrt{2\pi n}\left(\frac{n}{e}\right)^n$

S9. $\sum a_n$ converge or not $\Rightarrow$ check $\lim_{n\to\infty}\frac{a_{n+1}}{a_n}$ or $\sqrt[n]{a_n}$

$\boxed{\text{S10}}$ Irreducible* tran/recu?:
- tran: no stat & lim dist
- recu
  - Null: No stat & lim
  - Positive: unique stat
    - aper: has lim
    - per: no lim dis

* Irreducible $\Rightarrow$ all tran, or all recurrent ($\xi^{(i)}_j, j\in\mathbb{Z}$)

S11. $E_Y(E(X|Y))=E(X)$ ↓ 不一样!!!

$\boxed{\text{S12}}$ PGF of Z_i & num of children in gene i, f_i

$G_k(s)=f_0(f_1(\cdots(f_{k-1}(s))\cdots))$, $G_{Z_1}(s)=f_0$

S13. If P is irreducible, than the period of all states in MC is the same, and it is called periodicity of MC

$\boxed{\text{S14}}$ Prove positive recu or not: $E_xT_x<\infty$?

$E_xT_x=\sum_{i=0}^{\infty}P_x(T_x>i)$, 但证明了非po recu后, 不代表$x$是null recu! 看 $\rho_{xx}=\sum_{i=1}^{\infty}P(T_x=i)$,

若 $\rho_{xx}=1$, null; $\rho_{xx}<1$, transient

S14. 有时解π_i, $\pi_i=1/E_iT_i$ 比 $\pi P=\pi$ 更快

S15. 判断regular: P中用+填充来算 P^k

S16. $G_{Z_n}(s)$: $G_{Z_n}(0)=P(Z_n=0)=Pr\{$Is extinct at Generation $n\}$ | $G_{Z_{n+1}}(s)=G_{Z_n}(G_X(s))$

S17. $E[Z_{n+1}]=G'_{Z_{n+1}}(1)=G'_{Z_n}(\underline{G_X(1)})G'_X(1)$

$=G'_{Z_n}(1)G'_X(1)=E[Z_n]E[X]$

$\Rightarrow E[Z_n]=(E[X])^n$, X: 1个体offspring r.v.

接CMT & Martingales: ⑨ 有用: $n\to\infty$, $P(T>n)\to 0$

⑩ Routing Matrix 无负数 ⑪ $e_iA=E_i(V_A)$

Then First-Step: $e_{iA}=\frac{1}{\lambda_i}+\sum_{j\neq i}r(i,j)e_{jA}$

⑫ $E(M_{n\wedge T})=E(M_{n\wedge T};T\leq n)+E(M_{n\wedge T};T>n)$.

⑬ 5.10与5.11不同! $E[M_{n\wedge T}]=E[M_{n\wedge T};T\leq n]+E[M_{n\wedge T};T>n]$ trick: $n\to\infty$, 且:
$\lim_{n\to\infty}P(T>n)=P(T=\infty)$.

Exp Dist: ① T = exponential(λ), CDF: $P(T\le t) = 1-e^{-\lambda t}$ for $t\ge 0$; PDF: $f_T(t)=\lambda e^{-\lambda t}$ $(t\ge 0)$

$E(T) = 1/\lambda$. $Var(T) = 1/\lambda^2$

② Property: a. $P(T>t+s \mid T>t) = P(T>s)$

b. S, T are exp r.v. with λ and μ, then $\min(S,T) \sim$ exponential$(\lambda+\mu)$

And $P(S<T) = \int_0^\infty f_S(s) P(T>s)\,ds = \frac{\lambda}{\lambda+\mu}$

③. $V = \min(T_1, \cdots, T_n)$ and I be the index of smallest one, $T_i \sim$ Exponential(λ_i)

$$\begin{cases} P(V>t) = \exp(-(\lambda_1+\cdots+\lambda_n)t) \\ P(I=i) = \lambda_i/(\lambda_1+\cdots+\lambda_n) \end{cases}$$

Poisson Dist and Process:

① Def of Poisson Dist: If $X \sim$ Poisson(λ), then $P(X=n) = e^{-\lambda}\frac{\lambda^n}{n!}$, $E(X)=Var(X)=\lambda$

② $X\sim$Poisson(λ), then $E(X(X-1)\cdots(X-k+1)) = \lambda^k$

③ If $X_i \sim$ Poisson(λ_i), then $\sum_{i=1}^n X_i \sim$ Poisson$(\sum_{i=1}^n \lambda_i)$

④. $N\sim$ poisson(μ), $M|N \sim$ Binomial(N, p), then $M \sim$ Poisson(μp)

⑤ Def of PP: $\{N(s), s\ge 0\}$ is a PP if: a. $N(0)=0$ b. $N(s+t)-N(t) =$ Poisson(λs) c. $N(t)$: 单增, 且 $N(t_1)-N(t_0), \cdots, N(t_n)-N(t_{n-1})$ are independent.

※ Non-Homo PP: $\boxed{b.}$ $N(t)-N(s) \sim$ Poisson$(\int_s^t \lambda(r)dr)$

Law of rare events: ① If n is large, then binomial$(n, \lambda/n) \approx$ Poisson(λ)

$P(X=k) = \binom{n}{k}\left(\frac{\lambda}{n}\right)^k\left(1-\frac{k}{n}\right)^{n-k}$

② If trial ϵ_i: $Pr\{\epsilon_i=1\}=p_i$, $S_n = \epsilon_1+\cdots+\epsilon_n$, $\mu=\sum_{i=1}^n p_i$

$Pr\{S_n=k\} = \sum^{(k)} \prod_{i=1}^n p_i^{x_i}(1-p_i)^{1-x_i}$. then

$\left| Pr\{S_n=k\} - \frac{\mu^k e^{-\mu}}{k!}\right| \le \sum_{i=1}^n p_i^2$

Construct PP: ① Let $\tau_i \sim$ Exponential(λ). $T_n = \tau_1+\cdots+\tau_n$

$T_0 = 0$, and define $N(s) = \max\{n: T_n \le s\}$

0 — τ_1 — τ_2 — τ_3 ; T_0, T_1, T_2, S, T_3 $\quad N(s)=2$ (P&K) ($T_i \to W_i$, $\tau_i \to S_i$ in

② $T_n \sim$ gamma(n, λ), PDF: $f_{T_n}(t) = \lambda e^{-\lambda t}\frac{(\lambda t)^{n-1}}{(n-1)!}$

CDF: $F_{T_n}(t) = 1-\sum_{k=0}^{n-1}\frac{(\lambda t)^k e^{-\lambda t}}{k!}$; $N(s)\sim$ Poisson(λs)

③ $N(t+s)-N(s)$, $t\ge 0$ is a rate λ PP independent of $N(r)$, $r\in[0,S]$. (Markov Property)

Compound PP: ① $Y_1, Y_2 \cdots$ be i.i.d. r.v. Let N be 独立非负 r.v.

$S = Y_1+\cdots+Y_N$ with $S=0$ if $N=0$. Then

a. If $E|Y_i|, E(N)<\infty$, then $E(S) = E(N)\cdot E(Y_i)$

b. If $E(Y_i^2), E(N^2)<\infty$, then $Var(S) = E(N)\cdot Var(Y_i) + Var(N)\left[E(Y_i)\right]^2$

c. If $N\sim$ Poisson(λ), then $Var(S) = \lambda E(Y_i^2)$

Uniform Dist and PP: ① 0 — W_1 (U_2) — W_2 (U_n) — ⋯ — W_n (U_3) — t

$U_i \sim$ Uniform$([0,t])$, W_i: $\{U_i\}$ with order. Then joint PDF:

$f_{W_1,\cdots,W_n}(w_1, \cdots, w_n) = n!\,t^{-n}$, $0<w_1<\cdots<w_n\le t$

②. Let $W_1, \cdots$, be the occurence time of PP with rate λ.

Conditioned on $N(t)=n$: $f_{W_1,\cdots,W_n|N(t)=n}(w_1,\cdots,w_n) = n!\,t^{-n}$, $0<w_1<\cdots<w_n\le t$

PP!!

*Has important application in evaluating certain symmetric functionals on

Transformations: ①: $N_j(t)$: # of i s.t. $i\le N(t)$ and $Y_i=j$

Then $N_j(t)$ are independent rate $\lambda p(Y_i=j)$ PP

② PP with λ. We keep a point at s with $p(s)$, then new PP is a non-homo PP with rate $\lambda p(s)$, $N(t)-N(s)$: Poisson with mean $\int_s^t \lambda\cdot p(r)dr$ PP (CDF)

③. Call that starts at s and end at t: $G(t-s)$. Then # of call in progress at time t: Expectation $= \int_{s=0}^t \lambda(1-G(t-s))ds$

$= \lambda\int_{r=0}^t (1-G(r))dr \xrightarrow{t\to\infty} \lambda\mu$, μ is the mean of G.

④. Superposition: $N_1(t)\cdots N_k(t)$ are 独立 PP with $\lambda_1, \cdots, \lambda_k$

Then $N_1(t)+\cdots+N_k(t)$ is a PP with rate $\lambda_1+\cdots+\lambda_k$

⑤. Conditioning: If $s<t$ and $0\le m\le n$, then:

$P(N(s)=m \mid N(t)=n) = \binom{n}{m}\left(\frac{s}{t}\right)^m\left(1-\frac{s}{t}\right)^{n-m}$ (use *)

Technique: ① $\int_0^\infty t e^{-tx}dx = 1$, t is a constant

② For conclusion with multi, prove $n=2$, then merge (generalize)

③ $\lim_{n\to\infty}\left(1-\frac{\lambda}{n}\right)^n = e^{-\lambda}$ ⑤ $\sum_{n=0}^\infty \frac{t^n}{n!} \to e^t$

④ $X\in \mathbb{Z}^+$, $E(X) = \sum_{k=1}^\infty P(X\ge k)$; but in continuous domai

⑤ $Cov(X,Y) = E(XY) - E(X)E(Y)$

⑥ Def mean of Z $(Z\ge 0)$: $\int_0^\infty P(Z>u)du$ (Expectation,

⑦ PGF of Exp Dist: $\phi_{N(t)}(s) = \sum_{n=0}^\infty P(N(t)=n)s^n = e^{\lambda t(s-1)}$

⑧. $Var(X) = Var(E(X|Y)) + E(Var(X|Y))$.

$Var(X|Y) = E[X^2|Y] - (E[X|Y])^2$; Var!!

⑨ $E(s^{N(t)}) \Rightarrow \phi_{N(t)}(s)$!! ⑩ Use E instead of

⑪ $\int_0^\infty t^a e^{-bt}dt = \frac{a!}{b^a}$ ☆

⑫ $E[T_1] = P(T_1<a)E[T_1|T_1<a] + P(T_1\ge a)E[T_1|T_1\ge a]$

⑬ Conditioned on $N(t)$, then points in $[0,t]$ can be maneuvered in 'throwing darts' approach

⑭ $Y\sim$ Geometric(p) $\Rightarrow P(Y=k) = (1-p)^{k-1}p$

⑮ $cov(X,Y) = cov(X, Y-X+X)$ ☆

☆⑯. $E_{X_t}(\cdots, X_t)$: Use PGF! $\phi(s) = E[s^{X_t}] = e^{-\lambda t(1-s)}$, $X_t \sim$ Poisson(λt)

CMC: $P(X_{t+s}=j \mid X_s=i,\cdots) = P(X_t=j \mid X_0=i)$

$\Rightarrow P_t(i,j) = P(X_t=j \mid X_0=i)$ $\boxed{D_1}$

T1. Chapman-Kolmogorov: $\sum_k P_s(i,k)P_t(k,j)=P_{s+t}(i,j)$

D2. $q(i,j)=\lim_{h\to 0}\frac{P_h(i,j)}{h}$ for $j\ne i$ jump rate $i\to j$

D3. $\lambda_i=\sum_{j\ne i} q(i,j)$ is the rate X_t leaves i.

Then $r(i,j)=q(i,j)/\lambda_i$ prob of to j when leaving i

D4. $Q(i,j)=\begin{cases} q(i,j) & \text{if } j\ne i \\ -\lambda_i & \text{if } j=i \end{cases}$ ☆ D5. X_t is irreducible if $\forall i,j,\ P(i\to j)>0$

L1. If X_t is irreducible and $t>0$ then $P_t(i,j)>0$

☆. π is a stationary distribution iff $\pi Q=0$

T2. If a CMC X_t is irreducible and has π: $\lim_{t\to\infty} P_t(i,j)=\pi(j)$

☆ T3. If $\pi(k)q(k,j)=\pi(j)q(j,k)$, π is a stationary distribution

☆ Birth & Death chains. $S=\{0,1,\cdots,N\}$, $N\le\infty$.

$q(n,n+1)=\lambda_n$ $n<N$, $q(n,n-1)=\mu_n$ $n>0$, then: $\pi(n)=\frac{\lambda_{n-1}}{\mu_n}\pi(n-1)$

△ Exit Distribution and Exit Times $X_s\ne k$ for $s<t$

Let $V_k=\min\{t\ge 0: X_t=k\}$, $T_k=\min\{t\ge 0: X_t=k$ and

T4. Let $V_D=\min\{t: X_t\in D\}$, $T=\min\{V_A,V_B\}$. Suppose $C=S-(A\cup B)$ is finite and $P_x(T<\infty)>0$ for $x\in C$. If $h(a)=1$ for $a\in A$ and $h(b)=0$ for $b\in B$, and $\sum_j Q(i,j)h(j)=0$, then $h(i)=P_i(V_A<V_B)$.

⇒如何求出在$i\in C$中，先到A的概率！

T5. Suppose $C=S-A$ is finite and $P_i(V_A<\infty)>0$ for each $i\in C$. If $g(a)=0$ for $a\in A$, and for $i\in C=S-A$: $\sum_j Q(i,j)g(j)=-1$, then $g(i)=E_iV_A$

⇒求出从i出发到达A状态的时间期望

Martingales: D1. Indicator: $\mathbb{1}_A=\begin{cases}1: & x\in A\\ 0: & x\in A^c\end{cases}$

D2: $E(Y;A)=E(Y\cdot\mathbb{1}_A)\Rightarrow E(Y|A)=E(Y;A)/P(A)$

L1: $E(\phi(X)|A)\ge\phi(E(X|A))$

D3: Thinking M_n as the # of money at time n for a gambler at a fair game, and X_n as the outcomes, we say that $M_0, M_1,\cdots$ is a martingales w.r.t $X_0, X_1,\cdots$ if: (i) for any $n\ge 0$, $E|M_n|<\infty$

(ii) M_n can be determined from $X_n,\cdots,X_0$ and M_0

(iii) For any possible values $x_n,\cdots,x_0$:

$E(M_{n+1}-M_n \mid X_n=x_n,\cdots,X_0=x_0, M_0=m_0)=0$

D4. Supermartingales: $E(M_{n+1}-M_n|A_v)\le 0$

D5. Submartingales: $E(M_{n+1}-M_n|A_v)\ge 0$

D6. Quadratic $EX_i=0$, $Var(X_i)=\sigma^2$, then: $M_n=S_n^2-n\sigma^2$

D7. Exponential: Y_i: i.i.d distributed with $\phi(\theta)=E(\exp\theta Y_i)<\infty$. $S_n=S_0+Y_1+\cdots+Y_n$. Then $M_n=\exp(\theta S_n)/\phi(\theta)^n$

T1. X_n be a MC with transition prob p, let $f(x,n)$ be a func of state x and time n s.t.:

☆ $f(x,n)\ge\sum_y p(x,y)f(y,n+1)$.

Then $M_n=f(X_n,n)$ is a supermartingale w.r.t. X_n.

T2. If M_n is a supermartingale and $m\le n$, $EM_m\ge EM_n$

T3. M_n is a supermartingale w.r.t. X_n, H_n is predictable and $0\le H_n\le C_n$, C_n is a constant depend on n, then $W_n=W_0+\sum_{m=1}^{n}H_m(M_m-M_{m+1})$ is a supermartingale

D8. T: stopping time w.r.t. n. E.g. $H_m=\begin{cases}1, & \text{if } T\ge m\\ 0, & \text{else.}\end{cases}$

T4. (5.10) If M_n is a super w.r.t. X_n and T; then $M_{T\wedge n}$ is a super w.r.t. X_n, $EM_{T\wedge n}\le EM_0$ * or: $E(|M_{T\wedge n}|)\le K$

T5. (5.11) Suppose M_n is a super and [T a stopping finite time] 也是重要条件!! and $|M_{T\wedge n}|\le K$* for some constant K. Then $EM_T=EM_0$

⇒ In fair game, exit strategy won't change EM_T ($=EM_0$)

D9. Let X_n be a MC with S, $A,B\subset S$, $C-(A\cup B)$ is finite, Let $V_A=\min\{n\ge 0: X_n\in A\}$ and suppose that $P_x(V_A\wedge V_B<\infty)>0$ for all $x\in C$:

T6: If $h(a)=1$ for $a\in A$, $h(b)=0$ for $b\in B$, and for $x\in C$ we have $h(x)=\sum_y p(x,y)h(y)$

Then $h(x)=P_x(V_A<V_B)$

And $P_t'=P_tQ$ forward backward

① $P_t'(i,j)=\sum_{k\ne j}P_t(i,k)q(k,j)-P_t(i,j)\lambda_j$ [$P_t'=QP_t$]

$=\sum_{k\ne i}q(i,k)P_t(k,j)-\lambda_iP_t(i,j)$.

② $Q=\begin{pmatrix}-\lambda & \lambda\\ -\mu & \mu\end{pmatrix}\Rightarrow\begin{cases}P_t(1,1)=\frac{\mu}{\mu+\lambda}+\frac{\lambda}{\lambda+\mu}e^{-(\mu+\lambda)t}\\ P_t(2,1)=\frac{\mu}{\mu+\lambda}-\frac{\mu}{\lambda+\mu}e^{-(\mu+\lambda)t}.\end{cases}$

③ $\{N(t)\}_{t\ge 0}$ be a PP of λ, CMT $X=\{X(t)\}_{t\ge 0}$ via $X_t=Y_{N(t)}$, Y_n is a discrete MT with P.

Then for X, $Q_{i,j}(i\ne j)=\lambda P_{i,j}$

④ 证Martingale也可去证 $E(M_{n+1}|X_1,\cdots,X_n)=M_n$

同时也要证 $E|M_n|<\infty$与M_n由$X_n,\cdots,X_0$与M_0决定

⑤ $E[f(T)]=E[f(T)\cdot\mathbb{1}\{T<\infty\}]+E[f(T)\cdot\mathbb{1}\{T=\infty\}]$

⑥ $P_t'(i,j)=\sum_{k\ne j}q(i,k)P_t(k,j)-\lambda_iP_t(i,j)$

$P_t'(i,j)=\sum_{k\ne j}P_t(i,k)q(k,j)-P_t(i,j)\lambda_j$

⑦ If M_n是鞅, ϕ is a convex, then $\phi(M_n)$ is a sub鞅.

$[E[\phi(M_{n+1})|A_v]]\ge\phi(E(M_{n+1}|A_v))=\phi(M_n)$

⑧ 若M_n是鞅,则$E(M_{n+1}^2|A_v)-M_n^2=E((M_{n+1}-M_n)^2|A_v)$

且若$0\le i\le j\le k<n$, 则 $E[(M_n-M_k)(M_j-M_i)]=0$, 且:

$E(M_n-M_0)^2=\sum_{k=1}^{n}E(M_k-M_{k-1})^2$