Lec4&5&6

Bayes's Rule with Extra Conditioning:

Theorem: $P(A \cap E) > 0$ & $P(B \cap E) > 0$, we have

$$P(A|B,E) = \frac{P(B|A,E)\,P(A|E)}{P(B|E)}$$

证：设函数：$\hat{P}(\cdot) = P(\cdot|E)$.

$$\hat{P}(A|B) = \frac{\hat{P}(B|A)\cdot\hat{P}(A)}{\hat{P}(B)} = \frac{P(B|A,E)\,P(A|E)}{P(B|E)}.$$

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LOTP with Extra Conditioning:

$A_1, \cdots, A_n$ be a partition of S. $P(A_i \cap E) > 0$ for all i

Then: $P(B|E) = \sum_{i=1}^{n} P(B|A_i, E)\, P(A_i|E)$.

也是 $\tilde{P}(\cdot) = P(\cdot|E)$ 函数.

Conditioning As A Problem-Solving Tool

Eg: 有 $k+1$ coins, flipped, i^{th} coin turn heads with $p = i/k$. A coin is randomly selected and is then repeatedly flipped. 若 n 次 heads, 则下一次 head 概率?

建模: C_i: 选中 i^{th} coin (C: Cognition)

F_n: n 次头 H: 再抛一次头

$$P(H|F_n) = \sum_{i=0}^{k} P(H|C_i, F_n)\, P(C_i|F_n)$$

$$= \sum_{i=0}^{k} \left(\frac{i}{k}\right) \cdot \frac{P(F_n|C_i)P(C_i)}{P(F_n)} = \sum_{i=0}^{k} \left(\frac{i}{k}\right) \cdot \frac{\left(\frac{i}{k}\right)^n \cdot \frac{1}{k+1}}{\sum_{j=0}^{k} \left(\frac{j}{k}\right)^n \cdot \frac{1}{k+1}}$$

$$= \frac{\sum_{i=0}^{k} \left(\frac{i}{k}\right)^{n+1}}{\sum_{j=0}^{k} \left(\frac{j}{k}\right)^n}$$, 当 $k >> 1$ (趋于连续):

$$\lim_{k \to \infty} \frac{1}{k} \sum_{i=0}^{k} \left(\frac{i}{k}\right)^{n+1} = \int_0^1 x^{n+1}\, dx = \frac{1}{n+2} x^{n+2} \Big|_0^1 = \frac{1}{n+2}$$

$\therefore k >> 1$ 时, $P(H|F_n) = \frac{n+1}{n+2}$

又例: 一种虫, 1/3 死, 1/3 死又生一, 1/3 死又生二

求虫 die out 概率?

Campus

解: First-step Analysis: 首步分析

event D: die out

B_i: 虫变成 i 个虫, $i=0,1,2$

$P(D|B_0)=1 \quad P(D|B_1)=P(D) \quad P(D|B_2)=P(D)^2$

$\therefore P(D) = \sum_{i=0}^{2} P(D|B_i)P(B_i) = \frac{1}{3}(P(D)+P(D)^2+1)$

$\therefore P(D)^2-2P(D)+1=0 \quad P(D)=1$

$\therefore$ 虫群必定灭亡!

又例: Two gamblers, A&B, make a sequence of dollar bets. A win: p, B: $q=1-p$. A有 i dollars 且B有 $N-i$ dollar (i.e. A&B钱总数不变,为 N).

求A胜概率 (i.e. B破产).

解: First-step Analysis

event: A_i: A有 i 元 $\quad P_i$: $P(A \text{ win}|A_i)$

建模很重要!

$P_0=0 \quad P_N=1$ 而当 $1\le i\le N-1$ 下:

$$P_i = P(A \text{ wins}|A_i) = P(A \text{ wins}|\text{win}, A_i)P(\text{win}|A_i) + P(A \text{ wins}|\text{lose}, A_i)P(\text{lose}|A_i)$$

(LOTP).

$\therefore P_i = pP_{i+1}+qP_{i-1}$ 解数列 $(1\le i\le N-1)$.

$pP_{i+1}-tP_i=(1-t)P_i-qP_{i-1}$, 欲: $\frac{p}{t}=\frac{1-t}{q}$.

则 $t^2-t+pq=0$ 同时, $p+q=1$

且边界: $P_0=0 \quad P_N=1$

$\therefore (P_{i+1}-P_i)p=(P_i-P_{i-1})q \qquad \frac{P_{i+1}-P_i}{P_i-P_{i-1}}=\frac{q}{p}$

$(P_1-P_0)+(P_2-P_1)+\cdots+(P_N-P_{N-1})=1$

$\therefore (P_1-P_0)\cdot\frac{1-(\frac{q}{p})^{N-1}}{1-\frac{q}{p}}=1=P_1$ 这样就可求 P_i (幂级数相关)

(※用离散中齐次方程也可解!)

(见补)

KOKUYO

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最终解有：$P_i = \begin{cases} \dfrac{1-(\frac{q}{p})^i}{1-(\frac{q}{p})^N}, & p \neq q \\ \dfrac{i}{N}, & p = q = 1/2 \end{cases}$

Monty Hall Problem：

三个门，两个后面是山羊，一个后面是车。一个人选一个门，然后主持人知道哪个门是车，他在另两个门中开一个门，门后必是山羊。那么你应该换门吗？

应该换！ ↓choose.

car	~~goat~~	goat	1/3
g	c	~~g~~	1/3
g	~~g~~	c	1/3.

可见，另一个门为车的概率是 2/3，故应换

虽然表面上看起来是 1/2，但事实上是 1/3 : 2/3

牢记贝叶斯的训练！先验 ≠ 后验！（$1/2 \neq 1/3$）

灵魂：Monty 开门传递了信息！