普通物理I PHYS1181 第9讲

The Rotation and Dynamics of Rigid Bodies

《刚体的转动》**熟练掌握和灵活运用：**

本节课：角速度矢量；质心；转动惯量；转动动能；转动定律；力矩；力矩的功；定轴转动中的转动动能定律；

下节课：角动量和冲量矩；角动量定理；角动量守恒定律。

A1. 刚体

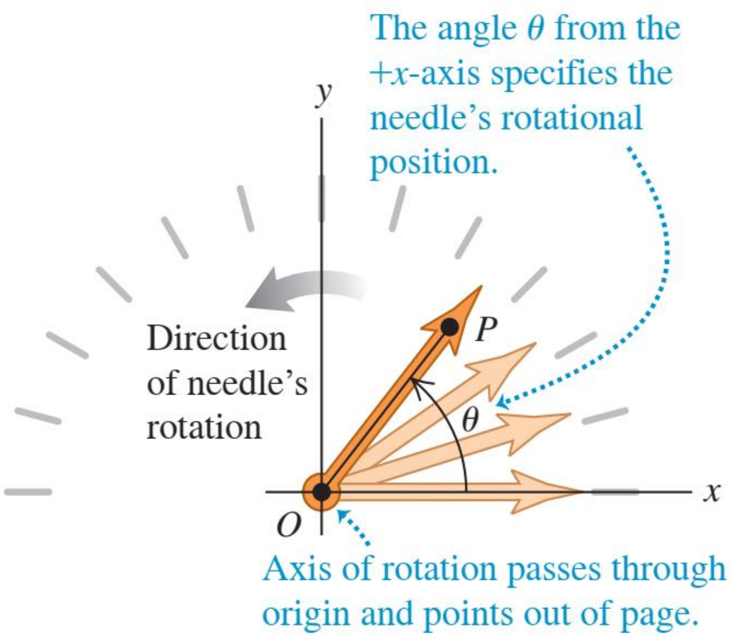
以刚体为研究对象，除了研究它的**平动**外，还研究它的**转动**以及**平动+转动**的复合运动等。

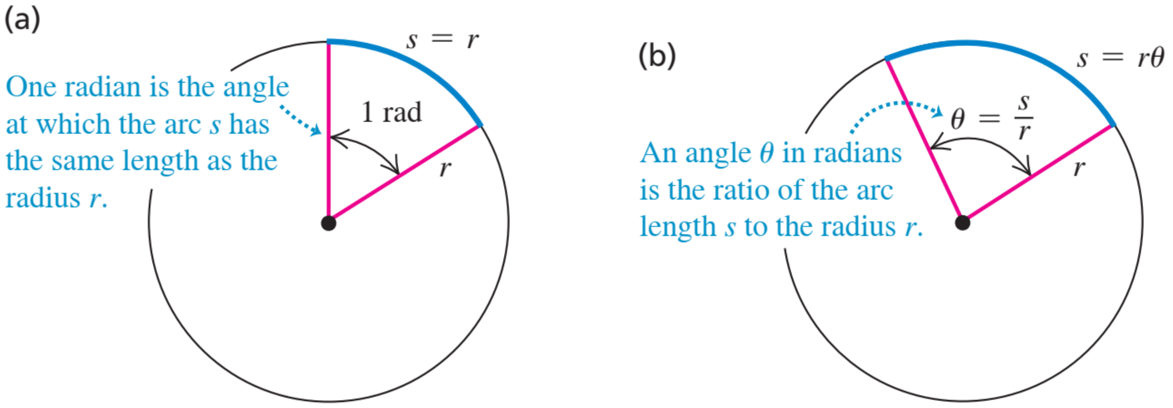
质点 —集合→ 质点系 —特例→ 刚体

质点 ↑ 不讨论物体的大小和形状

刚体 ↑ 讨论物体的大小和形状

B1. 角速度 angular velocity





$$\theta = \frac{s}{r} \quad \text{or} \quad s = r\theta \quad (\theta \text{ in radians})$$

定义弧度（radian，简写rad）：$1\ \text{rad} = \frac{360°}{2\pi} = 57.3°$

平均角速度：$\omega_{\text{av-}z} = \frac{\theta_2 - \theta_1}{t_2 - t_1} = \frac{\Delta\theta}{\Delta t}$

旋转刚体的每一部分都具有相同的平均角速度

思考： why? 轴在刚体外是否依然成立?

B2. 角加速度 angular acceleration

The **instantaneous angular acceleration** of a rigid body rotating around the z-axis ...

$$\alpha_z = \lim_{\Delta t \to 0} \frac{\Delta \omega_z}{\Delta t} = \frac{d\omega_z}{dt}$$

... equals the limit of the body's average angular acceleration as the time interval approaches zero ...

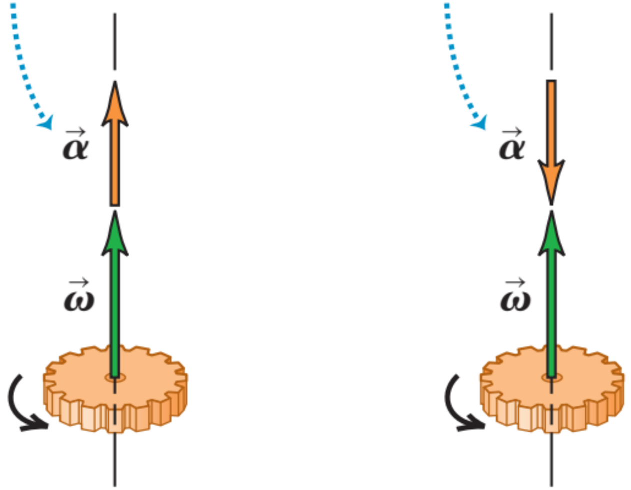
... and equals the instantaneous rate of change of the body's angular velocity.

$$\alpha_z = \frac{d}{dt}\frac{d\theta}{dt} = \frac{d^2\theta}{dt^2}$$

$$\vec{\alpha} = \frac{d\vec{\omega}}{dt}$$

$\vec{\alpha}$ and $\vec{\omega}$ in the **same** direction: Rotation speeding up.

$\vec{\alpha}$ and $\vec{\omega}$ in the **opposite** directions: Rotation slowing down.



当角速度方向不变，
只是大小改变时，角加速度沿着角速度的方向 → 完整满足向量操作

速度和角速度之间的关系 $\vec{v_t} = \vec{\omega} \times \vec{r}$

B3. 恒定角加速度转动

恒定加速度的直线平动 VS 恒定角加速度的固定轴转动

$$a_x = \text{constant}$$

$$v_x =$$

$$x =$$

$$v_x^2 =$$

$$x - x_0 =$$

$$\alpha_z = \text{constant}$$

$$\omega_z =$$

$$\theta =$$

$$\omega_z^2 =$$

$$\theta - \theta_0 =$$

B4．转动运动量和线性运动量的关系

线速度—角速度

Linear speed of a point on a rotating rigid body

$$v = r\omega$$

Angular speed of the rotating rigid body

Distance of that point from rotation axis

切向加速度—角加速度

Tangential acceleration of a point on a rotating rigid body

$$a_{\tan} = \frac{dv}{dt} = r\frac{d\omega}{dt} = r\alpha$$

Distance of that point from rotation axis

Rate of change of linear speed of that point

Rate of change of angular speed of body

径向加速度—角速度

Centripetal acceleration of a point on a rotating rigid body

$$a_{\mathrm{rad}} = \frac{v^2}{r} = \omega^2 r$$

Linear speed of that point

Angular speed of body

Distance of that point from rotation axis

C. 转动惯量 rotational inertia

考虑刚体中每一个质点的线速度： $v_i = r_i\omega$ 及其动能： $\frac{1}{2}m_i v_i^2 = \frac{1}{2}m_i r_i^2\omega^2$

则刚体转动的总动能： $K = \frac{1}{2}m_1 r_1^2\omega^2 + \frac{1}{2}m_2 r_2^2\omega^2 + \cdots = \sum_i \frac{1}{2}m_i r_i^2\omega^2$

如前所述，刚体中每个质点的角速度相等： $K = \frac{1}{2}(m_1 r_1^2 + m_2 r_2^2 + \cdots)\omega^2 = \frac{1}{2}\left(\sum_i m_i r_i^2\right)\omega^2$

定义： 刚体对于某一给定转动轴的惯性动量：

Moment of inertia of a body for a given rotation axis

Masses of the particles that make up the body

$$I = m_1 r_1^2 + m_2 r_2^2 + \cdots = \sum_i m_i r_i^2$$

Perpendicular distances of the particles from rotation axis

I is also called the *rotational inertia*，即 **转动惯量**。

D. 转动动能 rotational kinetic energy

定义： 围绕某一转动轴转动的刚体的**转动动能**：

Rotational kinetic energy of a rigid body rotating around an axis

$$K = \frac{1}{2} I \omega^2$$

Moment of inertia of body for given rotation axis

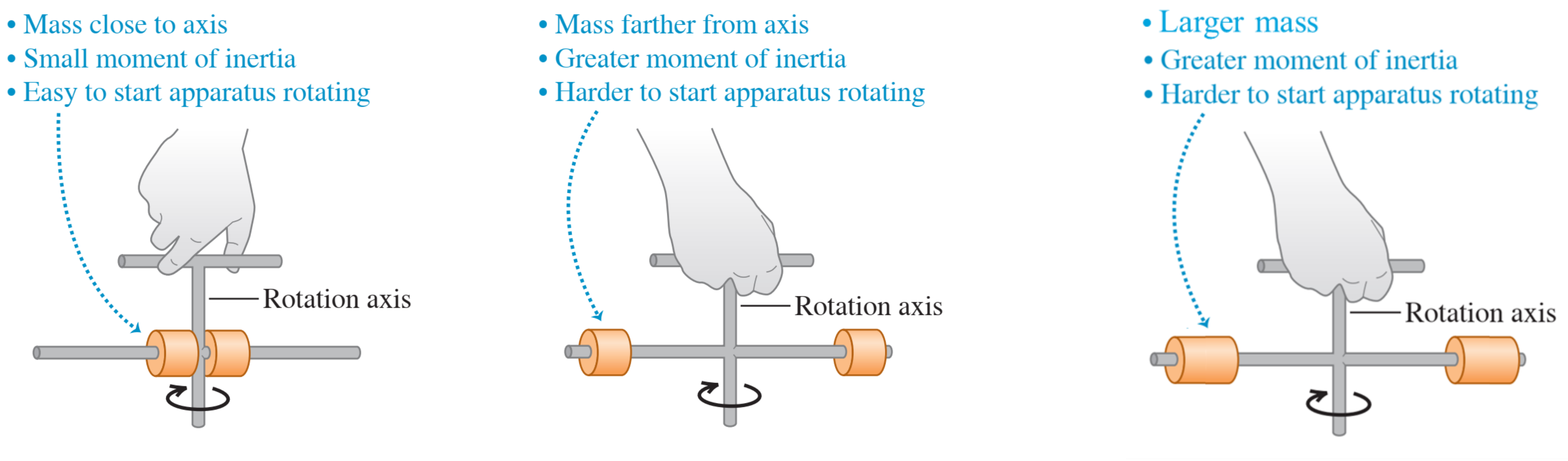
Angular speed of body

一个刚体的转动轴确定，转动惯量即确定，角速度越大，转动动能越大

不同的刚体，同样的角速度，转动惯量越大，转动动能越大。

对比：**平动动能**： $K = \dfrac{1}{2} m v^2$

E1．转动惯量的特点



$$I = m_1 r_1^2 + m_2 r_2^2 + \cdots = \sum_i m_i r_i^2$$

与m， $\boldsymbol{r}$ 均相关

E2. 转动惯量的计算

$$\rho = dm/dV$$

$$I = \int r^2\,dm = \int r^2 \rho\,dV = \rho \int r^2\,dV = \begin{cases} \int r^2 \eta\,dl & \text{线分布} \\ \int r^2 \sigma\,dS & \text{面分布} \\ \int r^2 \rho\,dV & \text{体分布} \end{cases}$$

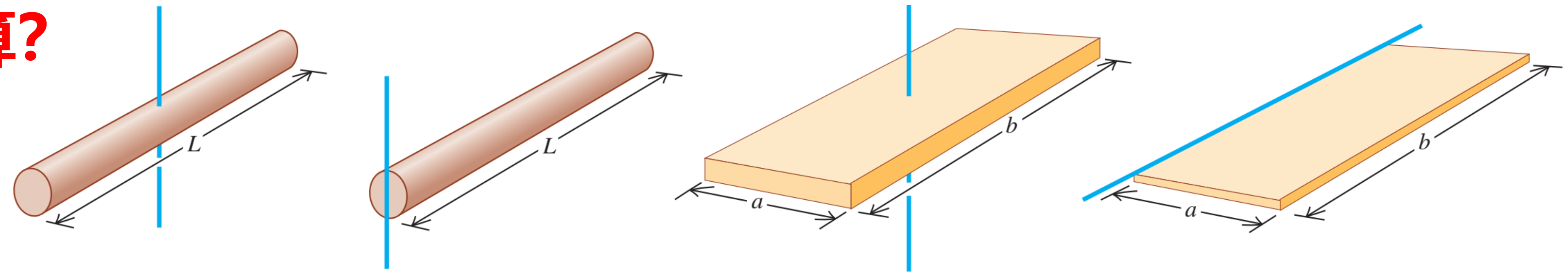
如何计算?

(a) Slender rod, axis through center

(b) Slender rod, axis through one end

(c) Rectangular plate, axis through center

(d) Thin rectangular plate, axis along edge



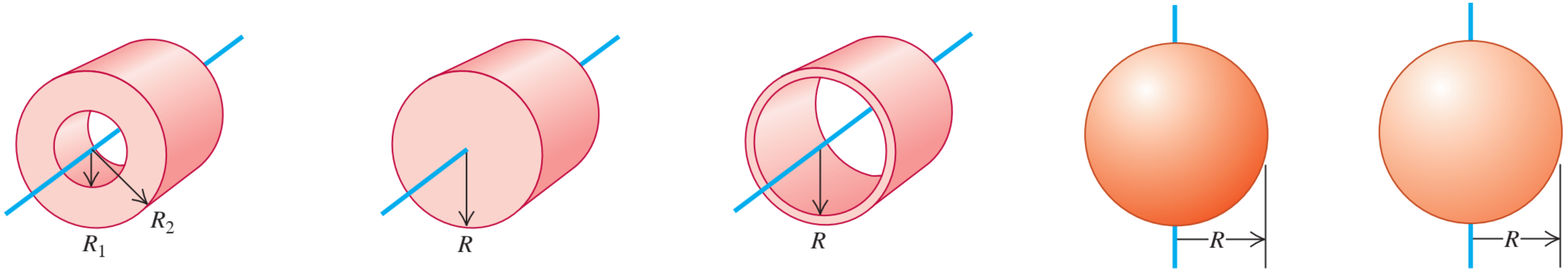
(e) Hollow cylinder

(f) Solid cylinder

(g) Thin-walled hollow cylinder

(h) Solid sphere

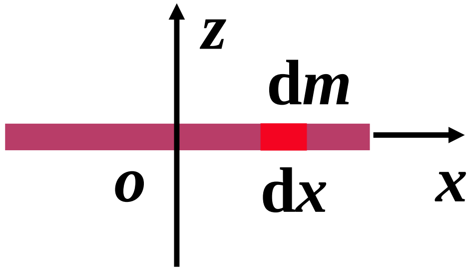
(i) Thin-walled hollow sphere



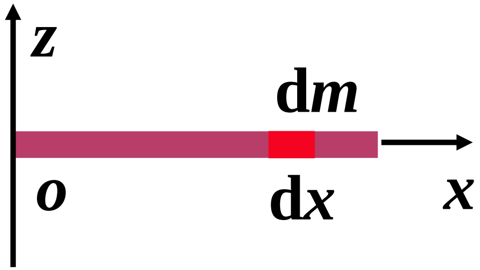
转动惯量 /和转轴有关，同一个物体对不同转轴的转动惯量是不同的

例如：均匀细棒的转动惯量：

a）转轴过中心与杆垂直



b）转轴过棒一端与棒垂直



E3．平行轴定理

设质量为 M 的刚体绕过质心cm的转轴的转动惯量为 I_{cm}； 绕过P点的转轴的转动惯量为 I_P； 两个转轴互相平行，相距为 d，则：

Parallel-axis theorem:

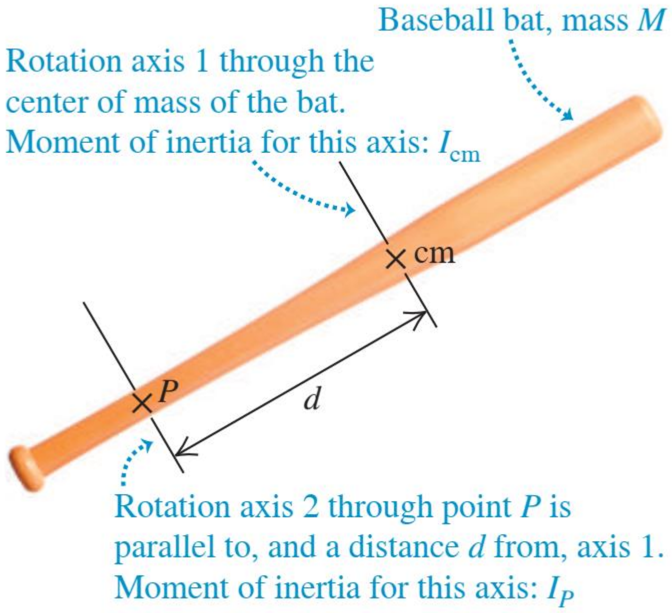
Moment of inertia of a body for a rotation axis through point P

$$I_P = I_{cm} + Md^2$$

Mass of body

Distance between two parallel axes

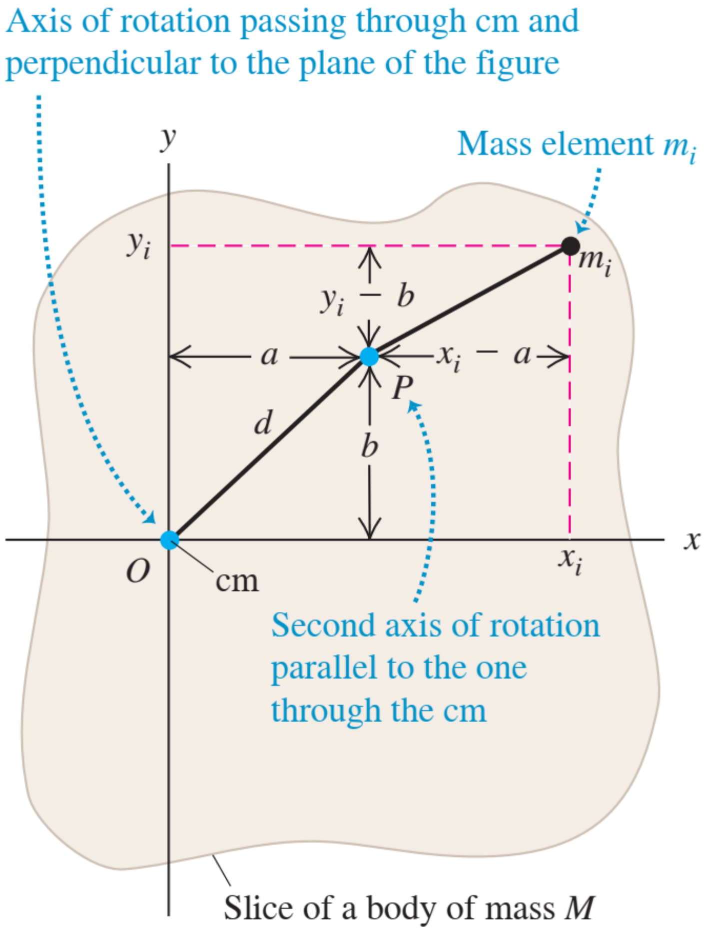
Moment of inertia of body for a parallel axis through center of mass



Parallel-axis theorem: $I_P = I_{cm} + Md^2$

注意：仅对质心轴 I_{cm}成立！

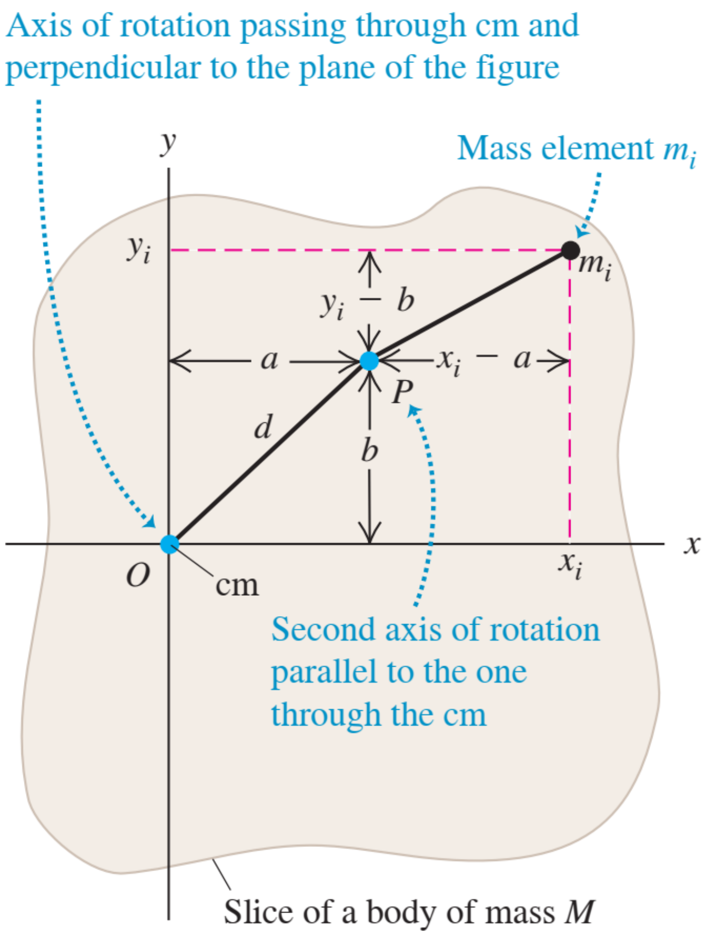
平行轴定理的简单证明



$$I_{cm} = \sum_i m_i ({x_i}^2 + {y_i}^2)$$

$$I_P = \sum_i m_i [(x_i - a)^2 + (y_i - b)^2]$$

$$I_P = \underbrace{\sum_i m_i ({x_i}^2 + {y_i}^2)}_{= I_{cm}} - \underbrace{2a \sum_i m_i x_i - 2b \sum_i m_i y_i}_{= 0} + \underbrace{(a^2 + b^2) \sum_i m_i}_{= Md^2}$$



$$I_P = \sum_i m_i(x_i^2 + y_i^2) - 2a\sum_i m_i x_i - 2b\sum_i m_i y_i + (a^2 + b^2)\sum_i m_i$$

$= I_{cm}$　　$= 0$　　$= 0$　　$= Md^2$

由于原点设在质心点，因此按照质心定义：

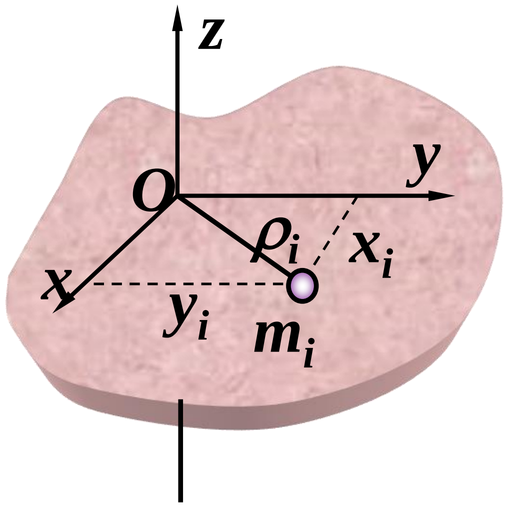
$$x_{\text{cm}} = \frac{m_1x_1 + m_2x_2 + m_3x_3 + \cdots}{m_1 + m_2 + m_3 + \cdots} = \frac{\sum_i m_i x_i}{\sum_i m_i} = 0$$

$$y_{\text{cm}} = \frac{m_1y_1 + m_2y_2 + m_3y_3 + \cdots}{m_1 + m_2 + m_3 + \cdots} = \frac{\sum_i m_i y_i}{\sum_i m_i} = 0$$

E4. 垂直轴定理 (薄片)

$$I_z = I_x + I_y$$

其中x，y为平面内正交的轴；z 为垂直平面的轴



E5. 可叠加定理

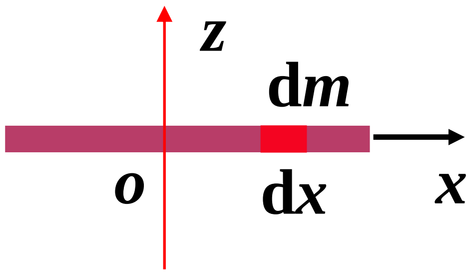
若一个复杂形状的物体是由许多简单形体组成，则这个复杂物体的对某轴的转动惯量等于各简单形体对同一转轴的转动惯量之叠加.

例1：

均匀细棒的转动惯量：

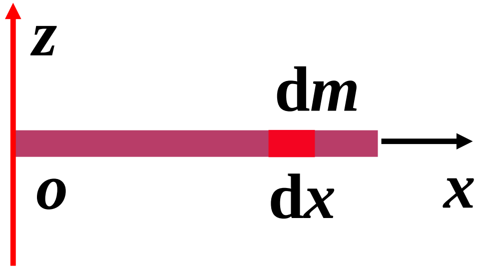
a）转轴过中心与杆垂直

$$I=\int r^2\mathrm{d}m=\int_{-\frac{l}{2}}^{\frac{l}{2}}x^2\frac{m}{l}\mathrm{d}x=\frac{1}{12}ml^2$$



b）转轴过棒一端与棒垂直

$$I=\int r^2\mathrm{d}m=\int_0^l x^2\frac{m}{l}\mathrm{d}x=\frac{1}{3}ml^2$$



或者可以直接应用平行轴定理得到：

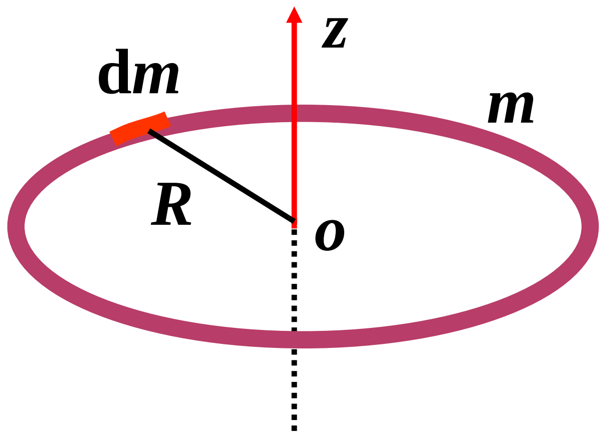
$$I=I_c+I_d=\frac{1}{12}ml^2+m\left(\frac{l}{2}\right)^2=\frac{1}{3}ml^2$$

例2：均匀细圆环的转动惯量

转轴过圆心与环面垂直

$$dm = \lambda \cdot dl \qquad \lambda = \frac{m}{2\pi R}$$

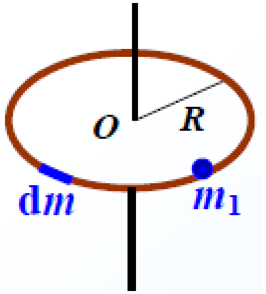
$$I = \int R^2 dm = \lambda R^2 \int_0^{2\pi R} dl = mR^2$$



思考1：圆环转轴通过圆环直径的转动惯量

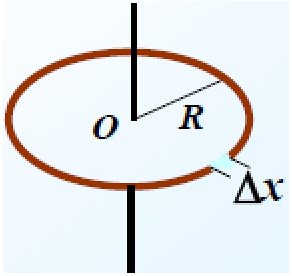
$$I_x = I_y = mR^2/2$$

思考2：圆环上加一质量为m_1质点，求I_z



$$I_z = mR^2 + m_1 R^2$$

思考3：圆环有一Δx的缺口，求I_z



$$I_z = mR^2 - \frac{m}{2\pi R}\Delta x R^2$$

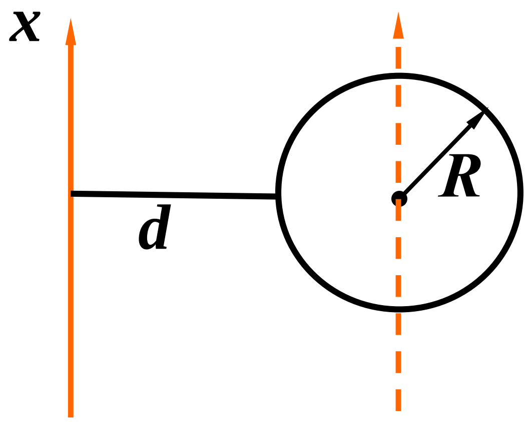
例3：如图,圆环质量m_1,半径R，短棒质量m_2，长度d，求对x轴的转动惯量

解：圆环转轴通过直径的转动惯量，根据垂直轴定理有

$$I_x = I_y = \frac{1}{2} I_z = \frac{1}{2} m_1 R^2$$

根据平行轴定理，圆环对转轴x的转动惯量为

$$I_1 = \frac{1}{2} m_1 R^2 + m_1 (R + d)^2$$

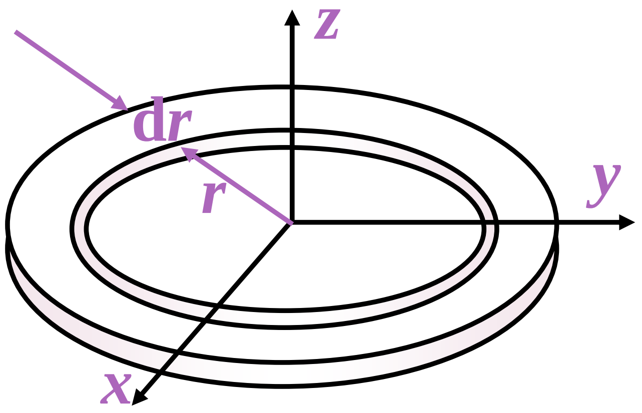


最后，根据叠加定理，整个元件对x轴的转动惯量为

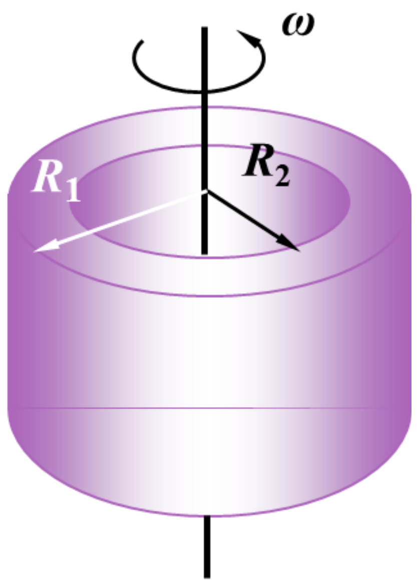
$$I = \frac{1}{3} m_2 d^2 + \frac{1}{2} m_1 R^2 + m_1 (R + d)^2$$

例4：均匀圆盘绕中心轴的转动惯量

质量为m, 半径为R的均匀圆盘, 转轴过圆心与圆盘垂直

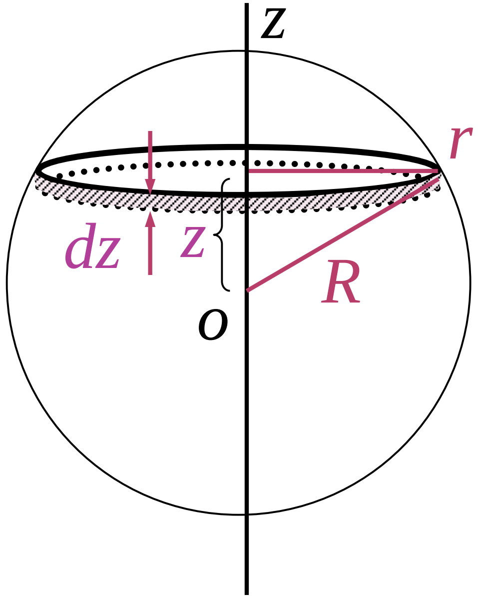


例5：求绕中心轴的空心圆柱的转动惯量



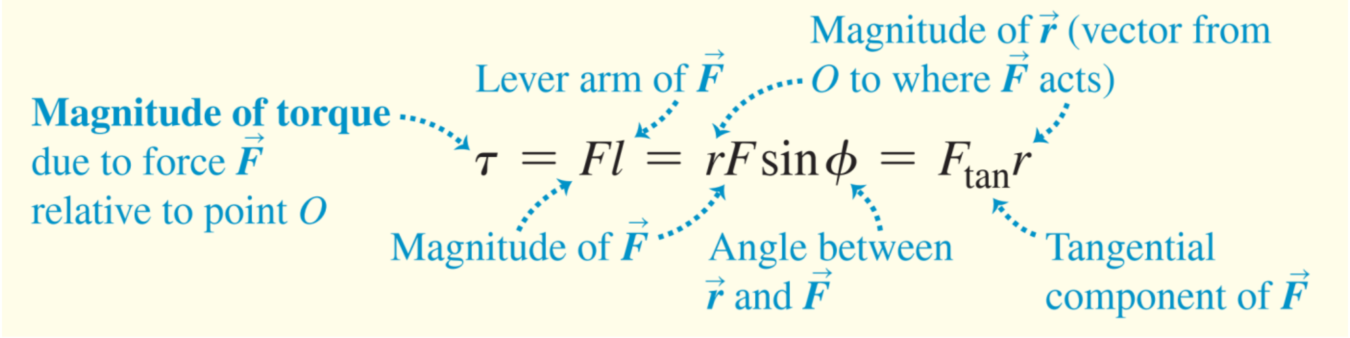
例6：求均匀球体绕直径的转动惯量

设球体的半径为R，总质量为m，密度为$\rho=3m/4\pi R^3$。



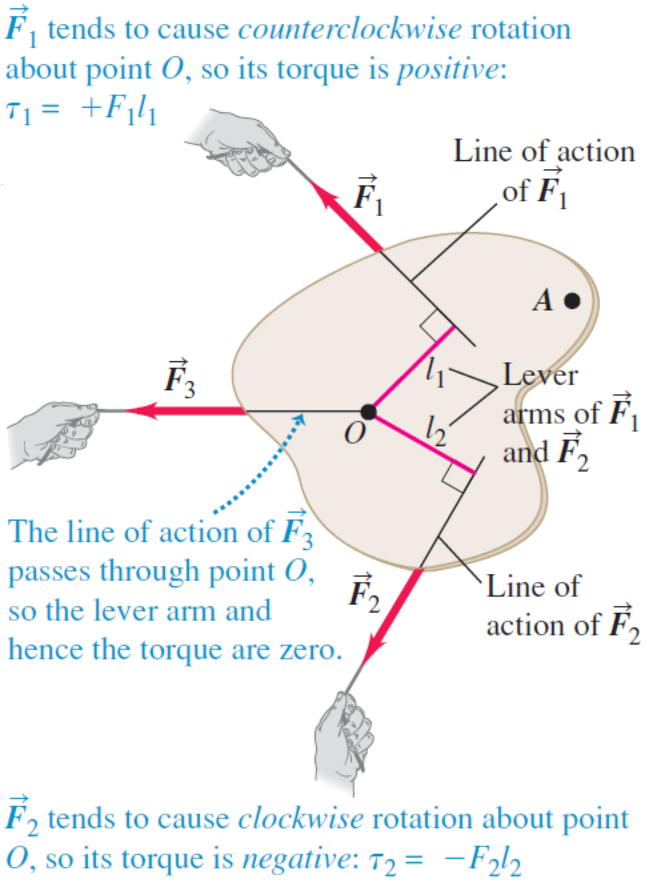
F. 力矩 （torque，又叫扭矩，moment）

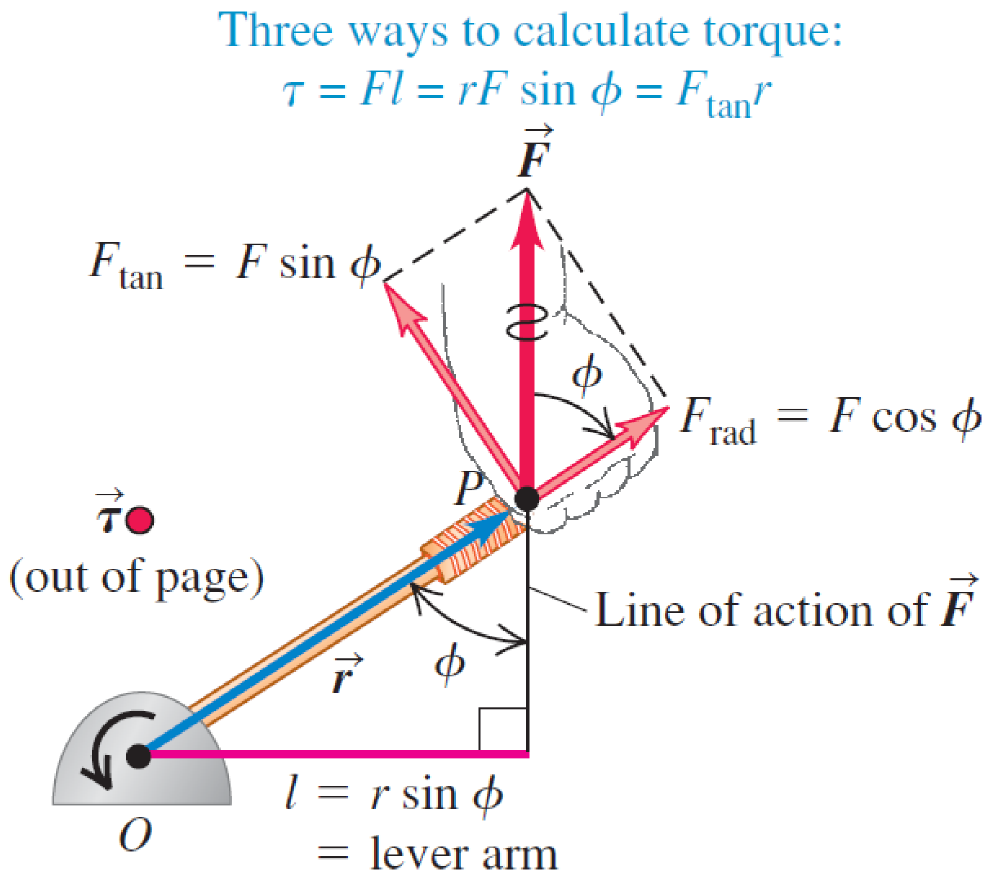
力矩的大小：



逆时针，正值：

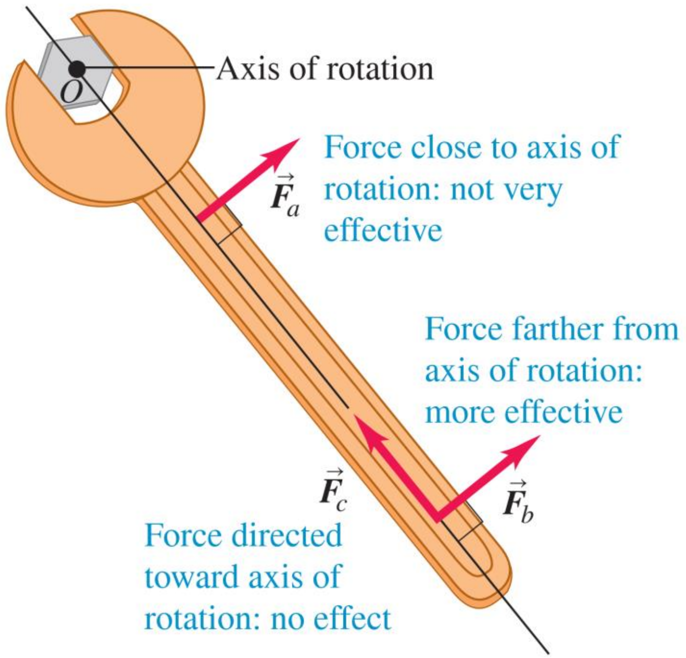
顺时针，负值：



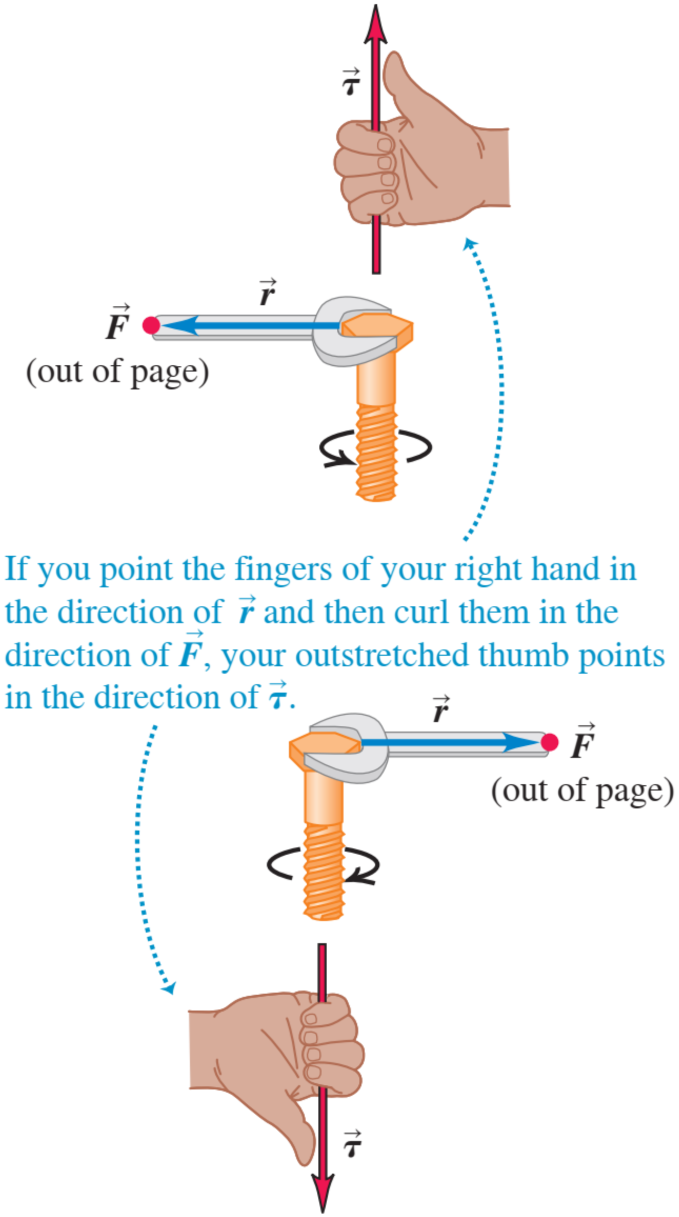


F. 力矩

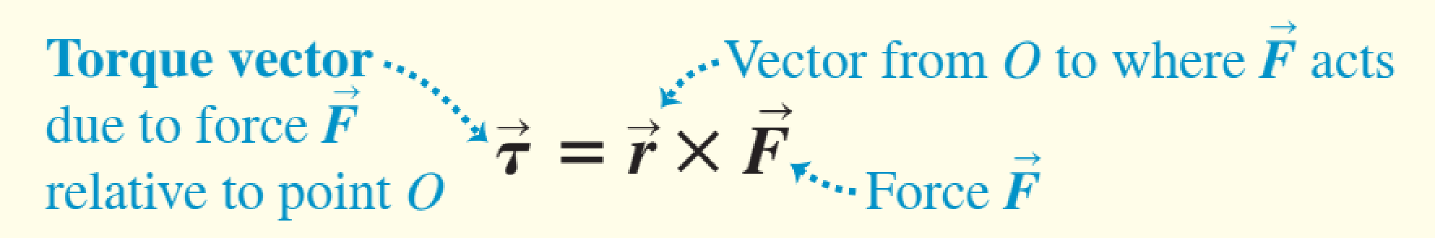
力矩的方向：



右手法则：



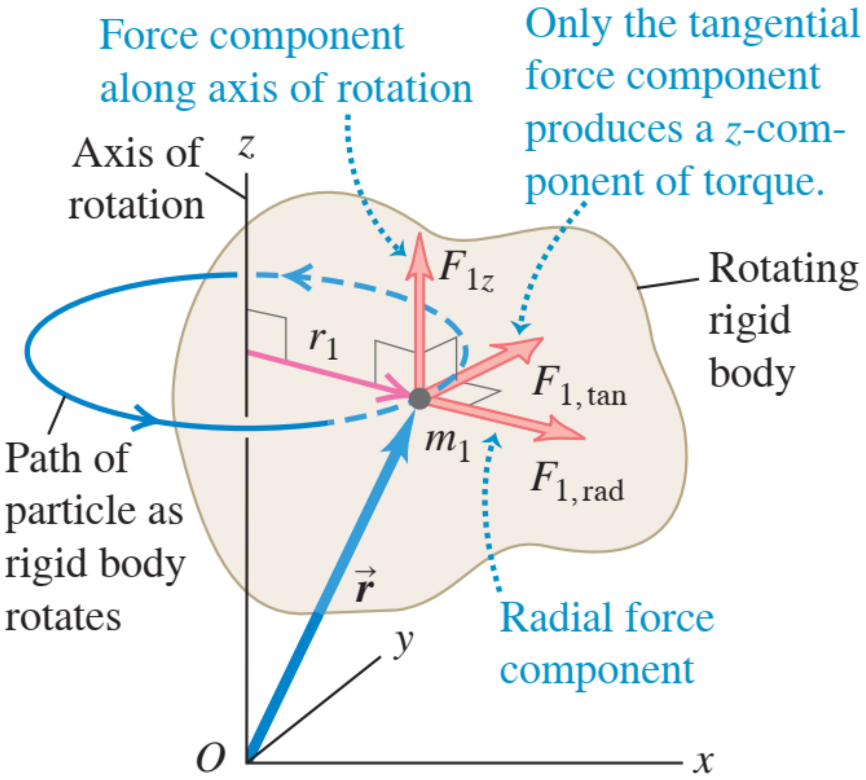
$\vec{F}$ 对参考点O的力矩为一矢量：



力矩是力臂和力的向量积

G．转动定律

刚体定轴转动定律：刚体在作定轴转动时,刚体的角加速度与它所受到的**合外力矩**成正比,与刚体的转动惯量成反比



Rotational analog of Newton's second law for a rigid body:

$$\sum \tau_z = I\alpha_z$$

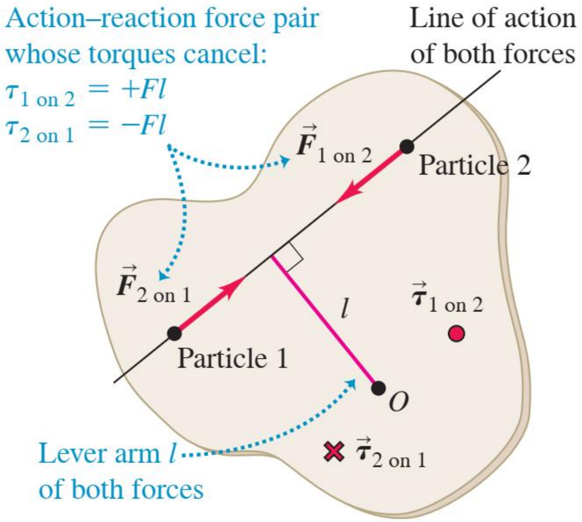
Net torque on a rigid body about z-axis

Moment of inertia of rigid body about z-axis

Angular acceleration of rigid body about z-axis

注意：合外力矩，转动惯量和角加速度都是相对于该转动轴的

合内力矩为零：

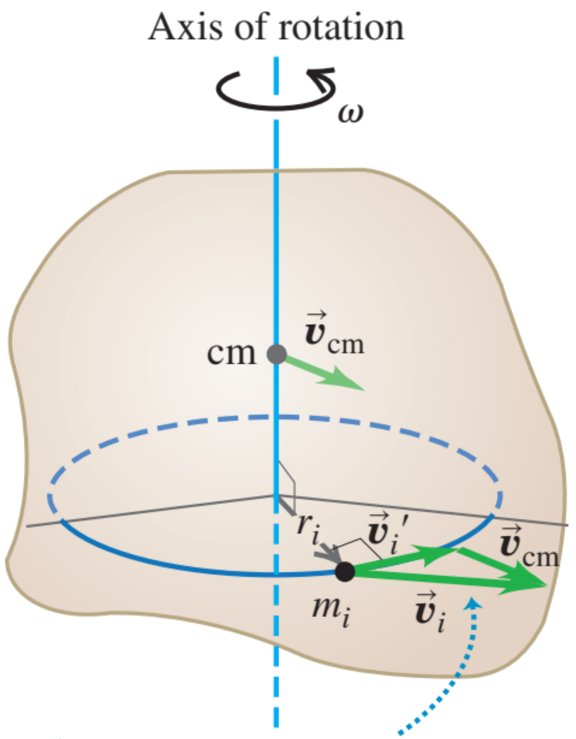


H-1. 刚体运动（平动+转动）的动能

刚体运动的组合规律：

刚体的任何运动都一定可以分解为**质心的平动**

+ 绕穿过质心的某一个轴的转动。



Velocity $\vec{v}_i$ of particle in rotating, translating rigid body = (velocity $\vec{v}_{cm}$ of center of mass) + (particle's velocity $\vec{v}_i{}'$ relative to center of mass)

$$\vec{v}_i = \vec{v}_{cm} + \vec{v}_i{}'$$

- $\vec{v}_i$：质点在惯性系中的速度
- $\vec{v}_{cm}$：质心速度
- $\vec{v}_i{}'$：质点相对于质心的速度

刚体运动的动能：

$$K = \frac{1}{2}Mv_{cm}^2 + \frac{1}{2}I_{cm}\omega^2$$

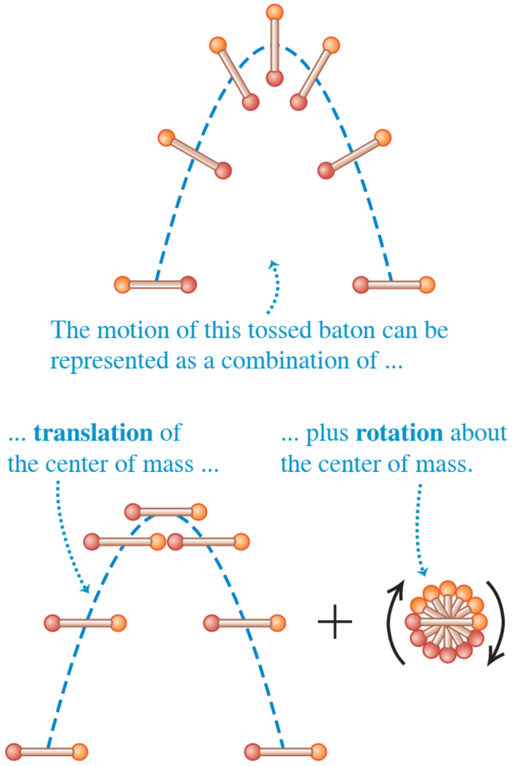
- **Kinetic energy of a rigid body with both translation and rotation**: K
- Kinetic energy of *translation* of center of mass (cm): $\frac{1}{2}Mv_{cm}^2$
- Kinetic energy of *rotation* around axis through cm: $\frac{1}{2}I_{cm}\omega^2$
- Mass of rigid body: M
- Speed of cm: v_{cm}
- Moment of inertia of rigid body about axis through cm: I_{cm}
- Angular speed of rigid body: ω

= 质心的动能 +围绕质心转动的转动动能

证明比较简单，同学们可以按上图分解成质点的动能之和来自己证明

刚体的任何运动都一定可以分解为**质心的平动 + 绕穿过质心的某一个轴的转动。**

The motion of a rigid body is a combination of translational motion of the center of mass and rotation around the center of mass.



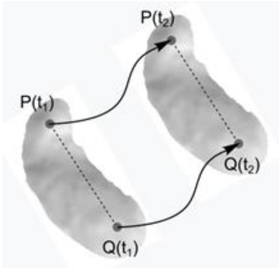
- 球棍质心的轨迹虽然是抛物线，但也是平移运动

刚体做圆周运动 ≠ 刚体转动！

- 摩天轮的小车轨迹虽然是圆周运动，但小车还是在做平移运动，而不是转动

如何理解平动?

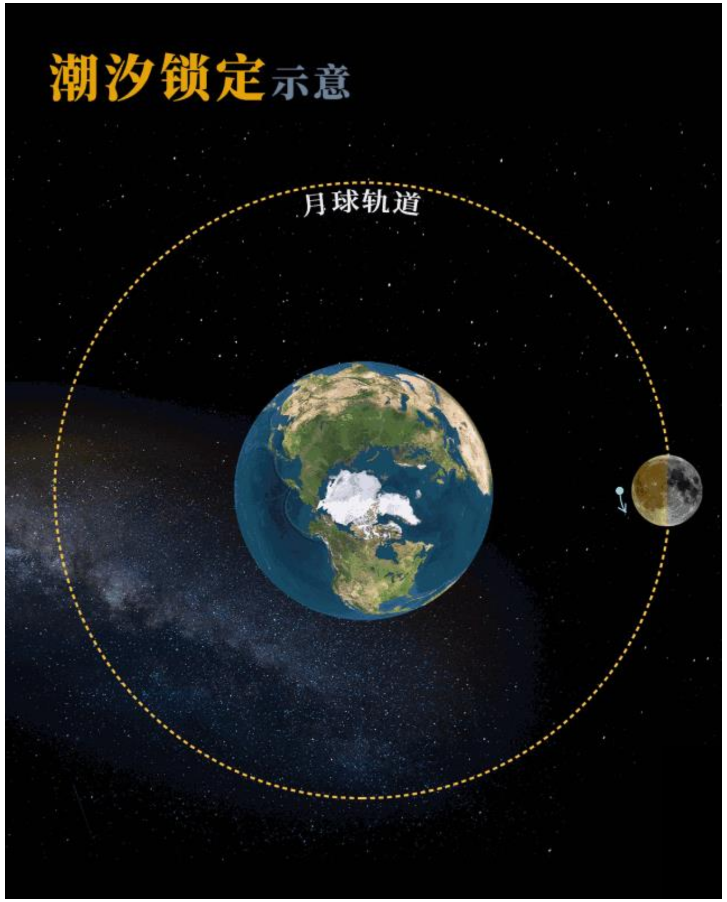
（平移运动，translational motion）



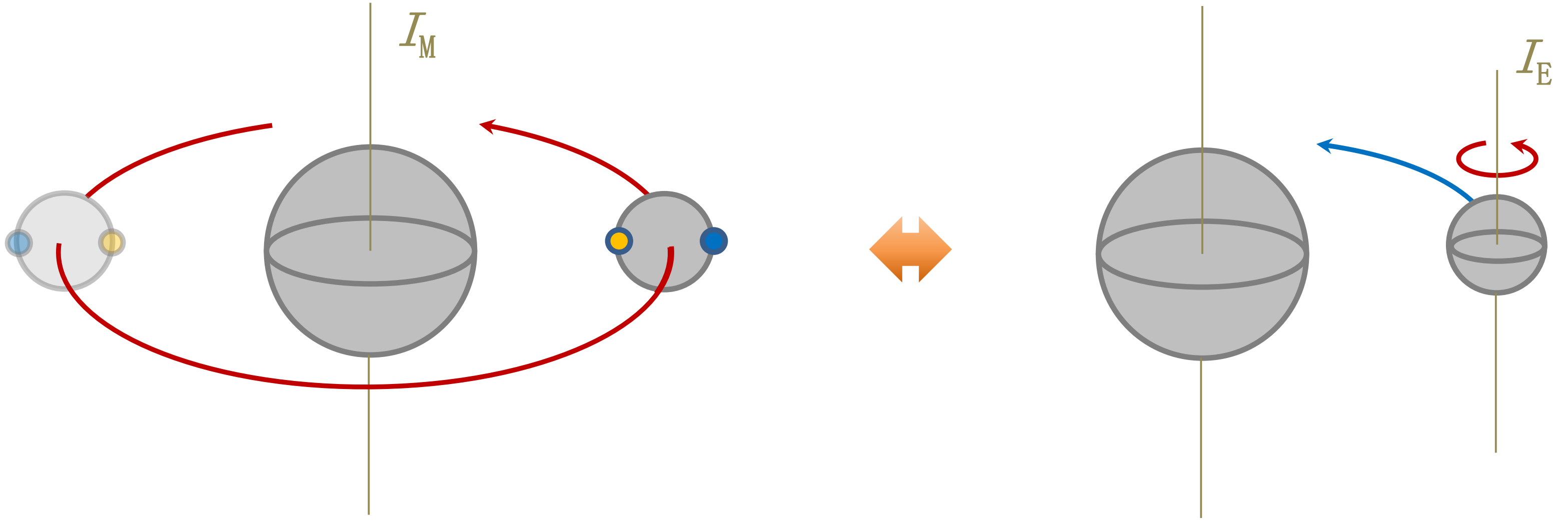
- ➢ 平移运动只针对刚体而言
- ➢ 平移运动的刚体，其内部所有质点的位移矢量、速度矢量、加速度矢量都相同
- ➢ 刚体的质心只有一个质点，因此刚体质心的运动就是平移运动，其他质点相对于刚体可以有转动

同时做圆周运动和转动的例子：同步自转

月亮的脸偷偷的在改变？错！月球永远只有一面朝向地球



月球的绕地轴的转动可分解为绕地轴的圆周平动+绕自身质心轴的转动



围绕地球轴的**转动** ⇔ 质心围绕地球轴的**圆周平动** + 围绕质心的**自转**

思考题：地球系中，求月球的公转转动惯量I_M 、自转转动惯量I_E 、公转动能K_E、自转动能K_M、总动能K、总势能U

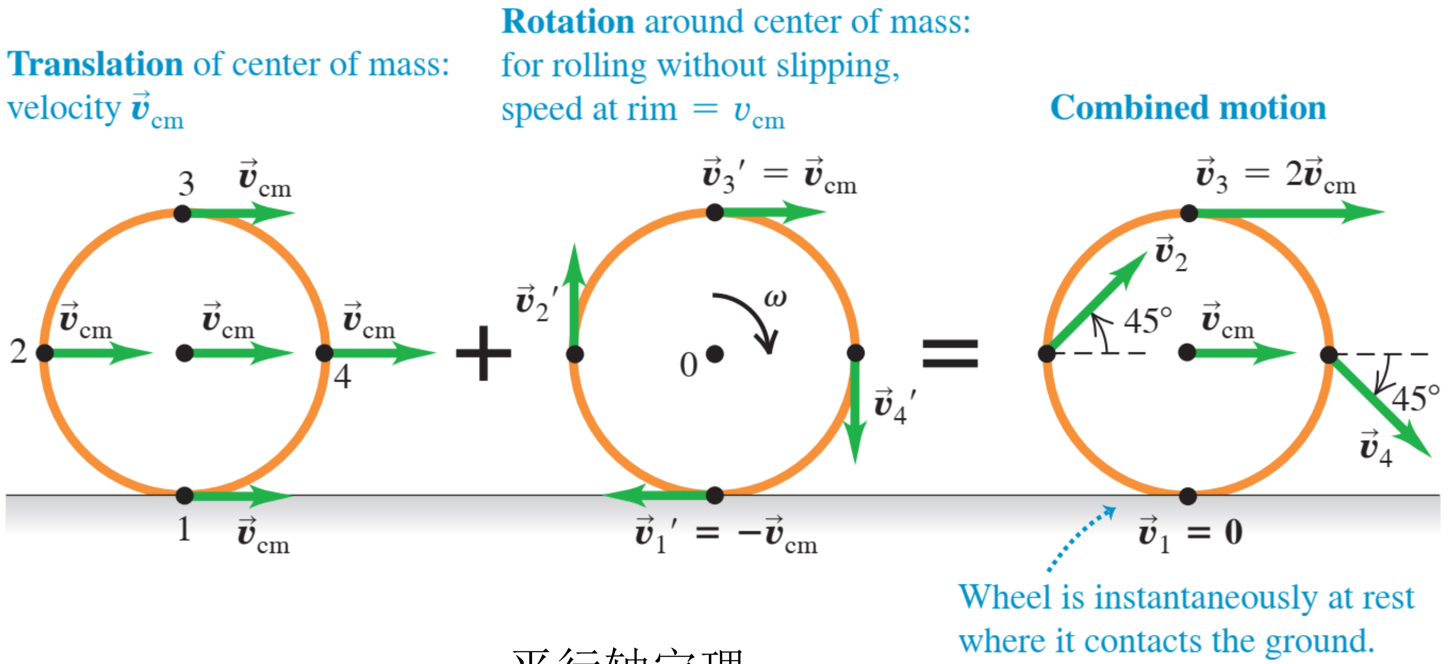
H-2. 无滑动的滚动

Condition for rolling without slipping:

$$v_{cm} = R\omega$$

Speed of center of mass of rolling wheel → v_{cm}; Radius of wheel → R; Angular speed of wheel → ω

各种车辆的车轮，只要不打滑，就是无滑动的滚动



平行轴定理

$$K = \tfrac{1}{2} I_1 \omega^2 \qquad I_1 = I_{cm} + MR^2$$

$$K = \tfrac{1}{2} I_1 \omega^2 = \tfrac{1}{2} I_{cm}\omega^2 + \tfrac{1}{2} MR^2\omega^2 = \tfrac{1}{2} I_{cm}\omega^2 + \tfrac{1}{2} M{v_{cm}}^2$$

生活中的实例：

汽车的时速表通过计算车轮转速和车轮半径来得到。出厂时按配备轮胎校准。

想一想：如果你改装你的爱车，换上更大号的轮胎，仪表盘显示的车速比实际车速更快还是更慢？

I-1. 力矩的功

力矩作的功等于力矩对角位置的积分

Work done by a torque τ_z:

$$W = \int_{\theta_1}^{\theta_2} \tau_z \, d\theta$$

Upper limit = final angular position

Integral of the torque with respect to angle

Lower limit = initial angular position

恒定力矩作的功等于力矩乘以角位移

Work done by a constant torque τ_z:

$$W = \tau_z(\theta_2 - \theta_1) = \tau_z \Delta\theta$$

Torque

Final minus initial angular position = angular displacement

力矩的功率等于力矩乘以角速度

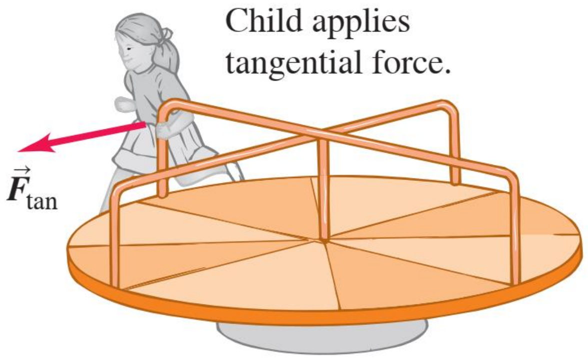
Power due to a torque acting on a rigid body:

$$P = \tau_z \omega_z$$

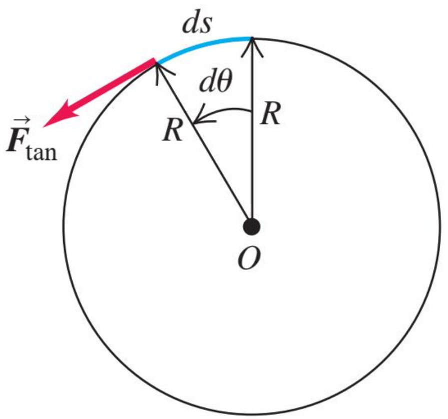
Torque with respect to rigid body's rotation axis

Angular velocity of rigid body about axis

(a)



(b) Overhead view of merry-go-round



$$dW = F_{tan} R \, d\theta$$

$$dW = \tau_z \, d\theta$$

I-2. 刚体定轴转动的动能定理

$$\tau_z \, d\theta = (I\alpha_z)\, d\theta = I\frac{d\omega_z}{dt} d\theta = I\frac{d\theta}{dt} d\omega_z = I\omega_z \, d\omega_z$$

刚体定轴转动的动能定理： （work-kinetic energy theorem）

Total work done on a rotating rigid body = work done by the net external torque

Final rotational kinetic energy

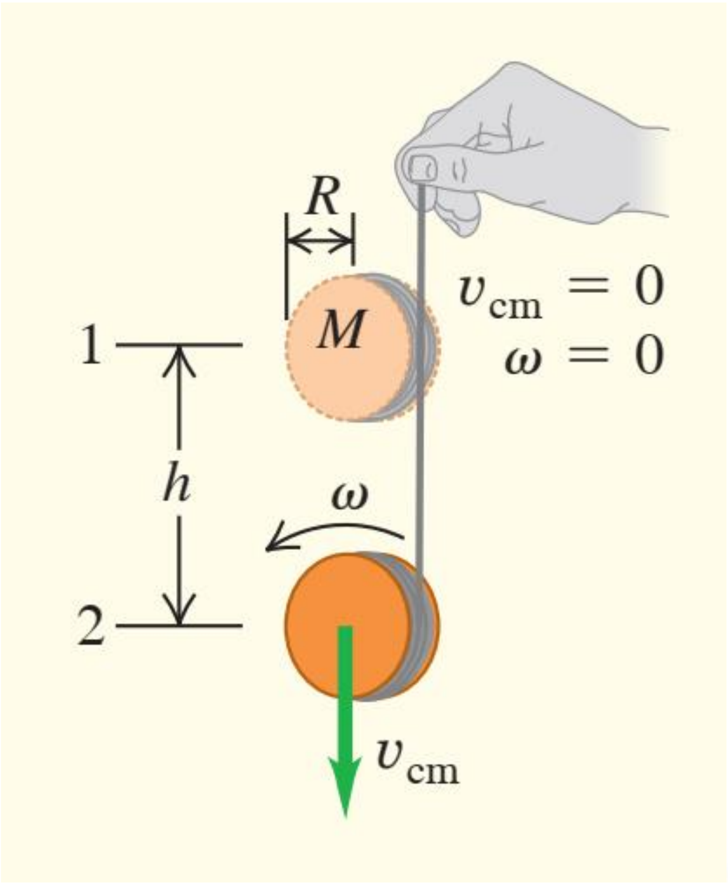
$$W_{\text{tot}} = \int_{\omega_1}^{\omega_2} I\omega_z \, d\omega_z = \tfrac{1}{2}I\omega_2^{\,2} - \tfrac{1}{2}I\omega_1^{\,2}$$

Initial rotational kinetic energy

合外力矩对刚体所作的功等于刚体转动动能的增量。

悠悠球

刚体的动能等于质心动能和相对于质心旋转的动能之和。



$$K_2 = \tfrac{1}{2}Mv_{cm}^{2} + \tfrac{1}{2}\left(\tfrac{1}{2}MR^2\right)\left(\frac{v_{cm}}{R}\right)^2 = \tfrac{3}{4}Mv_{cm}^{2}$$

$$K_1 + U_1 = K_2 + U_2$$

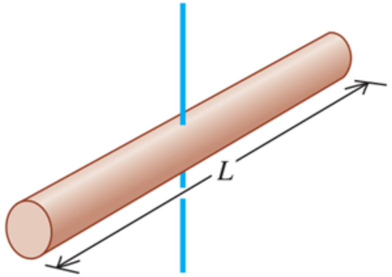
$$0 + Mgh = \tfrac{3}{4}Mv_{cm}^{2} + 0$$

$$v_{cm} = \sqrt{\tfrac{4}{3}gh}$$

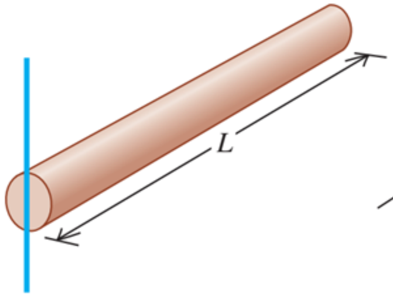
Homework 1

以下物体的质量均为M，求它们绕图中指示转轴的转动惯量，给出详细推导过程。

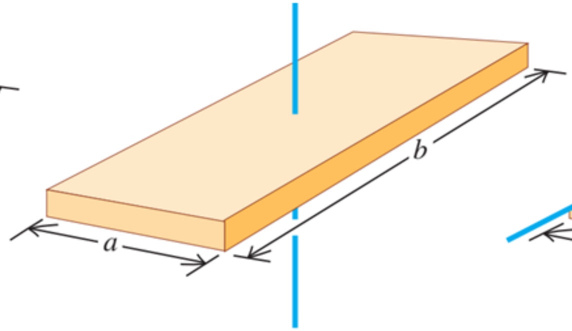
(a) Slender rod, axis through center



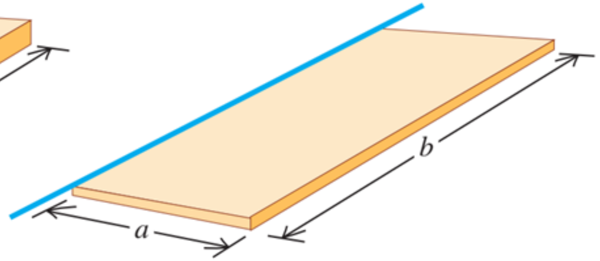
(b) Slender rod, axis through one end



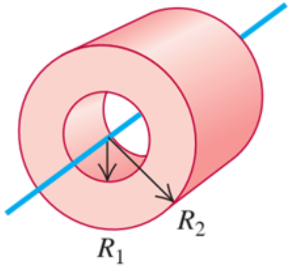
(c) Rectangular plate, axis through center



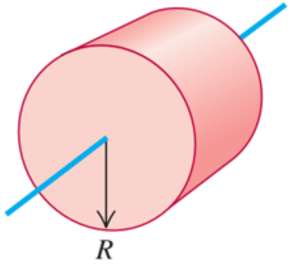
(d) Thin rectangular plate, axis along edge



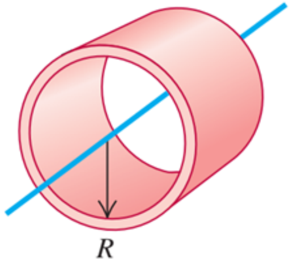
(e) Hollow cylinder



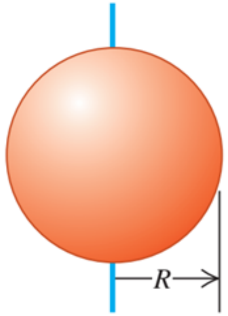
(f) Solid cylinder



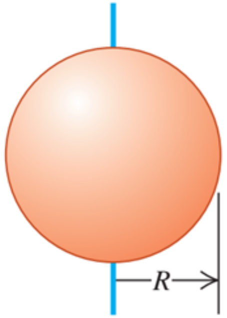
(g) Thin-walled hollow cylinder



(h) Solid sphere



(i) Thin-walled hollow sphere



Homework 2

一个哑铃由两个质量为m,半径为R的铁球和中间一根长l的连杆组成。和铁球的量相比,连杆的质量可以忽略。求此哑铃对于通过连杆中心并和它垂直的轴的转动惯量。它对于通过球的连心线的轴的转动惯量又是多大?