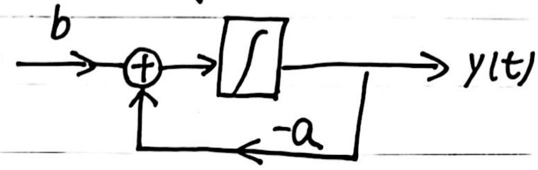
$*: \int_{-\infty}^{\infty} h(\tau) e^{s(t-\tau)} d\tau = e^{st}\int_{-\infty}^{\infty} h(\tau) e^{-s\tau} d\tau$

Chapter3 Lec 7&8&9

补充: $\frac{dy(t)}{dt}+ay(t)=bx(t)$ 的积分形式的Block Diagram:

$y(t)=\int_{-\infty}^{t}[bx(\tau)-ay(\tau)]d\tau$:



Chapter2中, $x[n]$要容易拆分为$\delta[n]$线性组合, 但若不容易呢?

The response of LTI systems to complex exponentials

连续: $e^{st} \longrightarrow LTI \longrightarrow \int_{-\infty}^{\infty} h(\tau) e^{-s\tau} d\tau \cdot e^{st} \overset{*}{=} H(s) e^{st}$.

此处e^{st}称为 an eigenfunction of the system

对于 specific value s, $H(s)$ 称为: corresponding eigenvalue

同时可以预见到: if $x(t)=a_1e^{s_1t}+a_2e^{s_2t}+a_3e^{s_3t}$

则$y(t)=a_1H(s_1)e^{s_1t}+a_2H(s_2)e^{s_2t}+a_1H(s_3)e^{s_3t}$

即: $x(t)=\sum_k a_k e^{s_kt} \Rightarrow y(t)=\sum_k a_k H(s_k) e^{s_kt}$

离散: $\underbrace{z^n}_{\text{eigenfunction}} \longrightarrow LTI \longrightarrow y[n]=?$

$y[n]=\sum_{k=-\infty}^{\infty} h[k] z^{n-k} = z^n \sum_{k=-\infty}^{\infty} h[k] z^{-k}$

令$H[z]=\sum_{k=-\infty}^{\infty} h[k]z^{-k}$, 则 $y[n]=\underline{H[z]}z^n$

for specific value z, $H[z]$ is the corresponding eigenvalue

也有: $x[n]=\sum_k a_k z_k^n \quad y[n]=\sum_k a_k H(z_k) z_k^n$

Summarize: $H(s)=\int_{-\infty}^{\infty} h(\tau) e^{-s\tau} d\tau \quad H[z]=\sum_{k=-\infty}^{\infty} h[k] z^{-k}$

$e^{st} \rightarrow LTI \rightarrow H(s)\, e^{st} \qquad z^n \rightarrow LTI \rightarrow H[z]z^n$

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Fourier series representation of continuous periodic signals

能把 $x(t)$ 表示为 complex exponentials 的线性组合么？

△: Harmonically related complex exponentials

(consider e^{st} with s purely imaginary)

$$\phi_k(t) = e^{jk\omega_0 t} = e^{jk(\frac{2\pi}{T})t}, \quad k = 0, \pm 1, \pm 2 \cdots$$

$k \neq 0$，则频率为 $|k|\omega_0$，周期为 $\frac{2\pi}{|k|\omega_0}$

则 $x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} = \sum_{k=-\infty}^{\infty} a_k e^{jk(\frac{2\pi}{T})t}$

这种表达便是傅利叶级数表示; ω_0 is the fundamental frequency

Notation: 对于 $a_k e^{jk\omega_0 t}$,
- $k=0$: DC component
- $k=\pm 1$: fundamental (first harmonic) component
- $k=\pm N$: Nth harmonic components

Eg: $x(t) = \sum_{k=-3}^{3} a_k e^{jk2\pi t}$, $a_0 = 1$, $a_{\pm 1} = 1/4$, $a_{\pm 2} = 1/2$, $a_{\pm 3} = 1/3$

$$x(t) = 1 + \frac{1}{4}(e^{j2\pi t} + e^{-j2\pi t}) + \frac{1}{2}(e^{j4\pi t} + e^{-j4\pi t}) + \frac{1}{3}(e^{j6\pi t} + e^{-j6\pi t})$$

$$= 1 + \frac{1}{2}\cos 2\pi t + \cos 4\pi t + \frac{2}{3}\cos 6\pi t.$$

即: 负频率的系数是正频率的系数的共轭

Real Signal : $x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$

$$x^*(t) = \sum_{k=-\infty}^{\infty} a_k^* e^{-jk\omega_0 t} = \sum_{k=-\infty}^{\infty} a_{-k}^* e^{jk\omega_0 t}$$

则: $x(t) = x^*(t)$ 可推出: $a_k = a_{-k}^*$ or $a_k^* = a_{-k}$ (Conjugate symmetry)

因此: alternative form of Fourier Series for real signal

$$x(t) = a_0 + \sum_{k=1}^{\infty} [a_k e^{jk\omega_0 t} + a_{-k} e^{-jk\omega_0 t}]$$

$$= a_0 + \sum_{k=1}^{\infty} 2\mathrm{Re}[a_k e^{j\omega_0 t}] = a_0 + 2\sum_{k=1}^{\infty} A_k \cos(k\omega_0 t + \theta_k)$$

$$a_k = A_k e^{j\theta_k}$$

*: $\int_T$ 代表: $\int_b^a$，其中 $a-b=T$，即在任意T长度内积分

Determine the FS representation Lec 8   * $k\in\mathbb{Z}$，代表用 $\{e^{jk\omega_0 t}\}$ 基底去表述

如果对于 $x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$ * 考虑：($T=\frac{2\pi}{|k|\omega_0}$)

$$x(t)\cdot e^{-jn\omega_0 t} = \sum_{k=-\infty}^{\infty} a_k e^{j(k-n)\omega_0 t}$$

$$\int_0^T x(t) e^{-jn\omega_0 t} = \sum_{k=-\infty}^{\infty} a_k \Big[\underbrace{\int_0^T e^{j(k-n)\omega_0 t} dt}\Big]_{I}$$

$$I: \begin{cases} k=n: & \int_0^T 1dt = T \\ k\neq n: & \text{在一个T内对这个周期信号积分} \Rightarrow 0 \end{cases}$$

可见: $I = T\delta[k-n]$

$$\therefore \int_0^T x(t) e^{-jn\omega_0 t} = Ta_n, \quad a_n = \frac{1}{T}\int_0^T x(t) e^{-jn\omega_0 t} dt$$

i.e. $a_k = \frac{1}{T}\int_T x(t) e^{-jk\omega_0 t} dt$ * (notational convention)

我们规定: $x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$ Synthesis Equation

$a_k = \frac{1}{T}\int_T x(t) e^{-jk\omega_0 t} dt$ Analysis Equation

特别地: $a_0 = \frac{1}{T}\int_T x(t)dt$

a_k: Fourier Series coefficients or spectral coefficients of $x(t)$

△: 灵性理解: $x(t)$ 分解到 $\{\cdots, e^{-j\omega_0 t}, 1, e^{j\omega_0 t}, \cdots\}$ 这个 basis 上，各basis上分成的量是 $\frac{1}{T}\int_0^T x(t) e^{-jk\omega_0 t} dt$

Eg: 求 $x(t) = \sin\omega_0 t$ 的 FS 系数:

$$\sin\omega_0 t = \frac{1}{2j} e^{j\omega_0 t} - \frac{1}{2j} e^{-j\omega_0 t} \qquad \frac{1}{2j}\cdot j\cdot\sin\omega_0 t$$

$$\left(\frac{1}{2j}\cos\omega_0 t + \frac{1}{2j}\cdot j\cdot\sin\omega_0 t - \frac{1}{2j}\cos(-\omega_0 t) - \frac{1}{2j}j\sin(-\omega_0 t)\right)$$

$\therefore a_1 \frac{1}{2j} \quad a_{-1} = -\frac{1}{2j}, \ a_k = 0 \text{ for } k\neq\pm 1$

Eg: $x(t) = 1 + \sin\omega_0 t + 2\cos\omega_0 t + \cos\left(2\omega_0 t + \frac{\pi}{4}\right)$

$$= 1 + \frac{1}{2j}\left[e^{j\omega_0 t} - e^{-j\omega_0 t}\right] + \left[e^{j\omega_0 t} + e^{-j\omega_0 t}\right] + \frac{1}{2}\left(e^{j(2\omega_0 t + \frac{\pi}{4})} + e^{-j(2\omega_0 t + \frac{\pi}{4})}\right)$$

No. $\ast\ sinc(x) = \frac{\sin(x)}{x}$

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$= 1 + (1+\frac{1}{2j})e^{jw_0t} + (1-\frac{1}{2j})e^{-jw_0t} + \frac{1}{2}e^{j\pi/4}e^{j2w_0t} + \frac{1}{2}e^{-j\pi/4}e^{-j2w_0t}$

Eg: 方波: $x(t) = \begin{cases} 1, & |t|<T_1 \\ 0, & T_1<|t|<T/2. \end{cases}$

$a_0 = \frac{1}{T}\int_{-T_1}^{T_1} 1\cdot dt = \frac{2T_1}{T}$ （因为 $T_1<|t|<T/2$ 下为0）.

$a_k = \frac{1}{T}\int_{-T_1}^{T_1} e^{-jkw_0t}dt = -\frac{1}{jkw_0T} e^{-jkw_0t}\Big|_{-T_1}^{T_1}$

$= \frac{2\sin(kw_0T_1)}{kw_0T} = \frac{2T_1}{T}\frac{\sin(kw_0T_1)}{kw_0T_1}$ $(k\neq 0)$

$= \frac{2T_1}{T}\, sinc(kw_0T_1)$ *

Convergance Problem

Approximate Periodic signal $x(t)$ by $x_N(t)$: $[e^{-jNw_0t}, \dots, e^{jNw_0t}]$ 相加

$\sum_{k=-N}^{N} a_k e^{jkw_0t}$ （原来: $(-\infty,\infty)$, 现用了

近似有多好呢? $e_N(t) = x(t) - x_N(t) = x(t) - \sum_{k=-N}^{N} a_k e^{jkw_0t}$

$E_N = \int_T |e_N(t)|^2 dt$

If $x(t)$ has a FS, $\boxed{\text{then } N\to\infty \Rightarrow E_N\to 0}$ ☆

问题在于:

① a_k may be infinite

② Even a_k is finite, when $N\to\infty$, $x_N(t)$ may not converge to $x(t)$

那么如果收敛, 则要满足以下 two classes of conditions.

① Finite energy condition: 在T内, 能量有限 ☆

If $\int_T |x(t)|^2 dt < \infty$, $x(t)$ 可用FS表示

② Dirichlet condition:

(1) Absolutely integrable: $\int_T |x(t)| dt < \infty$ 绝对可积☆

Eg: $x(t) = 1/t$, $t\in(0,1]$, 它不绝对可积

(2) 在有限时间间隔间, $x(t)$ is of bounded variation;

在一个周期内仅有有限个最大值与最小值 ☆

* FS: Fourier Series

No.

Eg: $x(t) = \sin(\frac{2\pi}{t})$, $0<t\le 1$, 满足(1)但(2)不行 (不连续点).

(3). 在有限时间间隔中, 只有有限个 finite discontinuities

信号不应有无限的跳跃或不连续

Properties of continuous-time FS*

notation: $x(t) \xleftrightarrow{FS} a_k$ coefficients

to signify the paring of a periodic signal with its FS

① Linearity: $x(t) \xleftrightarrow{FS} a_k$ $\quad y(t) \xleftrightarrow{FS} b_k$

$$z(t) = Ax(t) + By(t) \xleftrightarrow{FS} c_k = Aa_k + Bb_k$$

② Time shifting: $x(t) \xleftrightarrow{FS} a_k$

$$\Rightarrow x(t-t_0) \xleftrightarrow{FS} e^{-jk\omega_0 t_0} a_k$$

Proof: $\frac{1}{T}\int_T x(t-t_0) e^{-jk\omega_0 t} dt$

$\underline{\underline{t-t_0 \to \tau}}$ $\frac{1}{T}\int_T x(\tau) e^{-jk\omega_0(\tau+t_0)} d\tau$

$= e^{-jk\omega_0 t_0} \frac{1}{T}\int_T x(\tau) e^{-jk\omega_0 \tau} d\tau = e^{-jk\omega_0 t_0} a_k$

③ Time reversal $x(t) \xleftrightarrow{FS} a_k$

$$\Rightarrow y(t) = x(-t) \xleftrightarrow{FS} b_k = a_{-k}$$

若 $x(t)$ 偶, 则 $a_{-k} = a_k$; 若 $x(t)$ 奇, $a_{-k} = -a_k$

④ Time scaling: $x(t) \xleftrightarrow{FS} a_k \Rightarrow y(t) = x(\alpha t) \xleftrightarrow{FS} b_k = a_k$

虽然 bk 系数诚然不变, 但 fundamental frepuency 变了!

$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$, $x(\alpha t) = \sum_{k=-\infty}^{\infty} a_k e^{jk(\alpha\omega_0)t}$

⑤ multiplication: $x(t) \xleftrightarrow{FS} a_k$, $y(t) \xleftrightarrow{FS} b_k$ $\Rightarrow z(t) = x(t)y(t) \xleftrightarrow{FS} h_k = \sum_{l=-\infty}^{\infty} a_l b_{k-l}$

这一点很容易认同!

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* 这就是 $x^*(t)$ 的定义式！要牢记。因为：$e^{j\theta}$ 共轭为 $e^{-j\theta}$，a_k 共轭为 a_k^*

⑥ Conjugation and conjugate symmetry

$$x(t) \xleftrightarrow{FS} a_k \Rightarrow z(t) = x^*(t) \xleftrightarrow{FS} b_k = a_{-k}^*$$

Proof: $x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} \quad x^*(t) = \sum_{k=-\infty}^{\infty} a_k^* e^{-jk\omega_0 t}$ *

$$= \sum_{m=-\infty}^{\infty} a_{-m}^* e^{jm\omega_0 t}$$

如果 $x(t)$ 是实信号，则 $a_k^* = a_{-k}$

实且偶，则 $a_k^* = a_k$ 　　且显然：$a_0 = 0$

实且奇，则 $a_k + a_k^* = 0 \Rightarrow$ i.e., a_k 纯虚，无实部！

⑦ differentiation and integration

$$x(t) \xleftrightarrow{FS} a_k = \begin{cases} dx(t)/dt \xleftrightarrow{FS} jk\omega_0 a_k \\ \int_{-\infty}^{t} x(\tau)d\tau \xleftrightarrow{FS} a_k / jk\omega_0 \end{cases}$$

⑧ Frequency shifting

$$x(t) \xleftrightarrow{FS} a_k \Rightarrow e^{jM\omega_0 t} x(t) \xleftrightarrow{FS} a_{k-M}$$

这一点也很容易证明！

⑨ Periodic convolution

$$\begin{matrix} x(t) \xleftrightarrow{FS} a_k \\ y(t) \xleftrightarrow{FS} b_k \end{matrix} \Rightarrow \int_T x(\tau) y(t-\tau) d\tau \xleftrightarrow{FS} T a_k b_k$$

Proof: $\int_T x(\tau)y(t-\tau)d\tau = \int_T \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0\tau} \sum_{m=-\infty}^{\infty} b_m e^{jm\omega_0(t-\tau)} d\tau$

$$= \int_T \sum_{k=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} a_k e^{jk\omega_0\tau} b_m e^{-jm\omega_0\tau} e^{jm\omega_0 t} d\tau$$

$$= \sum_{k=-\infty}^{\infty} a_k \boxed{\sum_{m=-\infty}^{\infty} e^{jm\omega_0 t} b_m \underbrace{\int_T e^{jk\omega_0\tau} e^{-jm\omega_0\tau} d\tau}_{T\delta[k-m]}}$$

对于这一项来说，仅 $m=k$ 下，$T\delta[k-m] = T$，其余均为0！

$= \sum_{k=-\infty}^{\infty} T a_k b_k e^{jk\omega_0 t}$, i.e., $\int_T x(\tau)y(t-\tau)d\tau \xleftrightarrow{FS} T a_k b_k$

⑩ Parseval's relation

$$\frac{1}{T}\int_T |x(t)|^2 dt = \sum_{k=-\infty}^{\infty}|a_k|^2$$

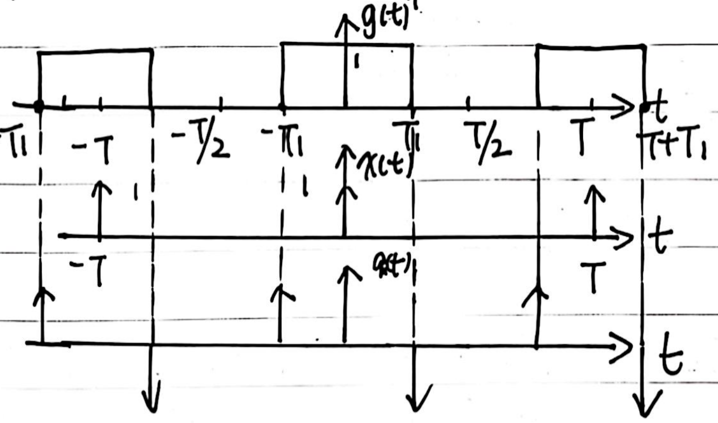
Proof: $\frac{1}{T}\int_T |x(t)|^2 dt = \frac{1}{T}\int_T x(t)x^*(t)dt = \frac{1}{T}\int_T x(t)\sum_{k=-\infty}^{\infty} a_k^* e^{-jk\omega_0 t}dt$

$= \sum_{k=-\infty}^{\infty} a_k^* \frac{1}{T}\int_T x(t)e^{-jk\omega_0 t}dt^{*} = \sum_{k=-\infty}^{\infty} a_k^* a_k = \sum_{k=-\infty}^{\infty}|a_k|^2$

*这一步，把Σ展开看看，就很容易认同这一步了

- $|a_k|^2$ is the average power in the k-th harmonic component of $x(t)$
- Total average power in $x(t)$ = the sum of average powers in all of its harmonic components

Example:



欲用FS表达g(t)，可用x(t) & g₁(t)，与Properties去求得！

$x(t)=\sum_{k=-\infty}^{\infty}\delta(t-kT)$，则 $a_k=\frac{1}{T}\int_{-T/2}^{T/2}\delta(t)e^{-jk\omega_0 t}dt=\frac{1}{T}$

$g_1(t)=x(t+T_1)-x(t-T_1)$（理解 → 左加右减）

$b_k = e^{jk\omega_0 T_1}a_k - e^{-jk\omega_0 T_1}a_k = \frac{1}{T}(e^{jk\omega_0 T_1}-e^{-jk\omega_0 T_1}) = \frac{2j\sin(k\omega_0 T_1)}{T}$

又 $g_1(t)=dg(t)/dt$，则 $b_k=jk\omega_0 c_k$

∴ $c_k=\frac{2j\sin(k\omega_0 T_1)}{jk\omega_0 T}=\frac{\sin(k\omega_0 T_1)}{k\pi}$ $(k\neq 0)$，$c_0=\frac{2T_1}{T}$ *

*: c_0可以用 $\lim_{k\to 0} c_k$ + 洛必达求.

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Lec10

Fourier series representation of discrete-time periodic signals

Harmonically related complex exponentials:

$$\phi_k[n] = e^{jk(2\pi/N)n}, \quad k=0,\pm 1,\pm 2.$$

则 fundamental frequency 为：$|k|\left(\frac{2\pi}{N}\right)$

注意：在$\phi_k[n]$中只有N个不同的信号，因为：$\phi_k[n]=\phi_{k+rN}[n]$

$$x[n]=\sum_{k=\langle N\rangle} a_k\phi_k[n]=\sum_{k=\langle N\rangle} a_k e^{jk\omega_0 n}=\sum_{k=\langle N\rangle} a_k e^{jk(2\pi/N)n}$$

那么a_k如何求？$x(t)$中，$a_k=\frac{1}{T}\int_T x(t)e^{-jk\omega_0 t}dt$.

则猜：$a_k=\frac{1}{N}\sum_{n=\langle N\rangle} x[n]e^{-jk(2\pi/N)n}$

证：$\sum_{n=\langle N\rangle} x[n]e^{-jr(2\pi/N)n}=\sum_{n=\langle N\rangle}\sum_{k=\langle N\rangle} a_k e^{jk(2\pi/N)n}e^{-jr(2\pi/N)n}$

$=\sum_{k=\langle N\rangle} a_k \boxed{\sum_{n=\langle N\rangle} e^{j(k-r)(2\pi/N)n}} \rightarrow = \begin{cases} N, & k=r \\ 0, & k\neq n \end{cases} = N\delta[k-r]$

$=Na_r$

$\therefore a_k=\frac{1}{N}\sum_{n=\langle N\rangle} x[n]e^{-jk(2\pi/N)n}$

由于$a_k=a_{k+rN}$，一般$x[n]$写作：$x[n]=a_0\phi_0[n]+\cdots+a_{N-1}\phi_{N-1}[n]=\sum_{k=\langle N\rangle}a_k\phi_k[n]$

但至于范围从几写到几是不固定的，只需保证$\langle N\rangle$即可

Eg: $x[n]=1+\sin\left(\frac{2\pi}{N}\right)n+3\cos\left(\frac{2\pi}{N}\right)n+\cos\left(\frac{4\pi}{N}n+\frac{\pi}{2}\right)$

$x[n]=1+\frac{1}{2j}[e^{j(2\pi/N)n}-e^{-j(2\pi/N)n}]+\frac{3}{2}[e^{j(2\pi/N)n}+e^{-j(2\pi/N)n}]$

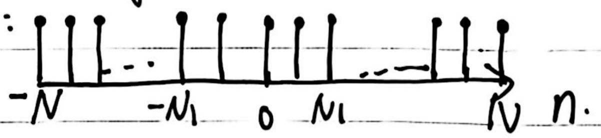
$+\frac{1}{2}[e^{j(4\pi n/N+\pi/2)}+e^{-j(4\pi n/N+\pi/2)}]$

$=\underbrace{1}_{a_0}+\underbrace{\left(\frac{3}{2}+\frac{1}{2j}\right)}_{a_1}e^{j(2\pi/N)n}+\underbrace{\left(\frac{3}{2}-\frac{1}{2j}\right)}_{a_{-1}}e^{-j(2\pi/N)n}$

$+\underbrace{\frac{1}{2}e^{j\pi/2}}_{a_2}e^{j2(2\pi/N)n}+\underbrace{\frac{1}{2}e^{-j\pi/2}}_{a_{-2}}e^{-j2(2\pi/N)n}$

Real Signal: $x[n] = x^*[n] \Rightarrow a_k = a_{-k}^*$

例:



$x[n]$:

$$a_k = \frac{1}{N}\sum_{n=-N_1}^{N_1} x[n]\, e^{-jk(2\pi/N)n}$$

$$= \frac{1}{N}\sum_{m}^{2N_1} e^{-jk(2\pi/N)(m-N_1)}$$

$$= \frac{1}{N} e^{jk(2\pi/N)N_1}\sum_{m=0}^{2N_1} e^{-jk(2\pi/N)m}$$

$k=0$ 时: $a_k = \dfrac{2N_1+1}{N}$

$(\pm N, \pm 2N)$

$k \neq 0, \pm N, \pm 2N$ 时: $a_k = \dfrac{1}{N} e^{jk(2\pi/N)N_1}\left(\dfrac{1-e^{-jk2\pi(2N_1+1)/N}}{1-e^{-jk(2\pi/N)}}\right)$

$$= \cdots = \frac{1}{N}\frac{\sin[2\pi k(N_1+1/2)/N]}{\sin(\pi k/N)}$$

Properties of FS (Discrete)

Property	$x[n], y[n]$: $\omega_0 = 2\pi/N$	FS coefficient
Linearity	$Ax[n] + By[n]$	$Aa_k + Bb_k$
Time Shifting	$x[n-n_0]$	$a_k e^{-jk(2\pi/N)n_0}$
Frequency Shifting	$e^{jM(2\pi/N)n} x[n]$	a_{k-M}
Conjugating	$x^*[n]$	a_{-k}^*
Time Reversal	$x[-n]$	a_{-k}
Time Scaling	$x_{(m)}[n] = \begin{cases} x[n/m], & \text{if } m \mid n \\ 0, & \text{if } m \nmid n. \end{cases}$	$\frac{1}{m} a_k$
Periodic Convolution	$\sum_{r=\langle N\rangle} x[r]\, y[n-r]$	$N a_k b_k$
Multiplication	$x[n]\, y[n]$	$\sum_{l=\langle N\rangle} a_l b_{k-l}$
First Difference.	$x[n] - x[n-1]$	$\left(1-e^{-jk(2\pi/N)}\right) a_k$
Running Sum	$\sum_{k=-\infty}^{n} x[k]$ (finite valued & periodic only if $a_0 = 0$)	$\left(\frac{1}{1-e^{-jk(2\pi/N)}}\right) a_k$

Date

Parseval's relation: $\frac{1}{N}\sum_{t=\langle N\rangle}|x[n]|^2=\sum_{t=\langle N\rangle}|a_k|^2$

☆: 对于离散FS来说，没有收敛问题！

FS & LTI System

$e^{st}\xrightarrow{LTI}H(s)e^{st}$ $\qquad z^n\xrightarrow{LTI}H[z]z^n.$

其中 $H(s)=\int_{-\infty}^{\infty}h(\tau)e^{-s\tau}d\tau$ $\qquad H[z]=\sum_{k=-\infty}^{\infty}h[k]z^{-k}$

而对于 jw 表达形式：

连 $e^{jwt}\xrightarrow{LTI}H(jw)e^{jwt}$ $\qquad H(\boxed{jw})=\int_{-\infty}^{\infty}h(\tau)e^{-jw\tau}d\tau$

则 $x(t)=\sum_{k=-\infty}^{\infty}a_ke^{jkw_0t}\xrightarrow{LTI}y(t)=\sum_{k=-\infty}^{\infty}a_kH(jkw_0)e^{jkw_0t}$

离 $H(\boxed{e^{jw}})=\sum_{n=-\infty}^{\infty}h[n]e^{-jwn}$ $\qquad$ ($z=e^{jw}$，因为 s,z 定义方式不同)

$e^{jwn}\xrightarrow{LTI}H(e^{jw})e^{jwn}$

$x[n]=\sum_{k=\langle N\rangle}a_ke^{jk(2\pi/N)n}\xrightarrow{LTI}y[n]=\sum_{k=\langle N\rangle}a_kH(e^{jk(2\pi/N)})e^{jk(2\pi/N)n}$

Date

Lec 11. (FT)

Chapter 4 非周期连续时间信号的Fourier Transformation

FT pair:

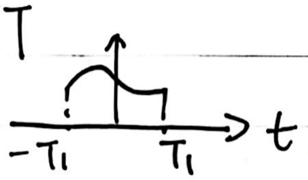
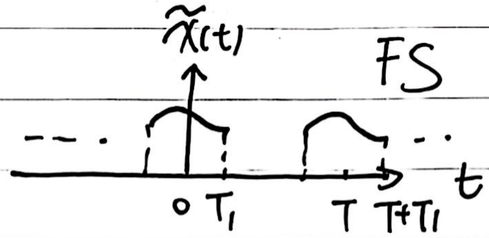
幅度因子: $X(j\omega) = \int_{-\infty}^{+\infty} x(t) e^{-j\omega t} dt$ FT

signal: $x(t) = \frac{1}{T}\int_{-\infty}^{+\infty} X(j\omega) e^{j\omega t} dt$ Inverse FT

$= \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(j\omega) e^{j\omega t} d\omega$

称: $X(j\omega)(d\frac{\omega}{2\pi})$ 是不同频率的权重

$X(j\omega)$ 称为 spectrum

FT  vs  FS

a signal of finite duration

$X(j\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt$ $a_k = \frac{1}{T}\int_T \tilde{x}(t) e^{-jk\omega_0 t} dt$

$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega) e^{j\omega t} d\omega$ $\tilde{x}(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$

FT收敛条件:

① $\int_{-\infty}^{\infty} |x(t)|^2 dt < \infty$

② $\int_{-\infty}^{\infty} |x(t)| dt < \infty$

③. 有限个最大值与最小值

④. 有限区间中只有有限个不连续点

Eg: $x(t) = e^{-at} u(t)$ $a > 0$

$X(j\omega) = \int_0^{\infty} e^{-at} e^{-j\omega t} dt = \frac{1}{a + j\omega}$

$x(t) = \delta(t)$, $X(j\omega) = \int_{-\infty}^{\infty} \delta(t) e^{-j\omega t} dt = 1$

$x(t) = 1$. $X(j\omega) = \int_{-\infty}^{\infty} e^{-j\omega t} dt = 2\pi\delta(\omega)$ *

$\omega \neq 0$下, 由于周期, 在$t \in [-\infty, +\infty]$下, $\int$必为0

* $\delta(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} 1 \cdot e^{j\omega t} d\omega$, $\delta(-\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} 1 \cdot e^{-j\omega t} dt$.

Eg. $x(t)=\begin{cases}1, & |t|<T_1 \\ 0, & |t|>T_1\end{cases}$ $X(jw)=\int_{-T_1}^{T_1} e^{-jwt} = \frac{2\sin wT_1}{w}$

Eg. $X(jw)=\begin{cases}1, & |w|<W \\ 0, & |w|>W\end{cases}$, $x(t)=?$

$x(t)=\frac{1}{2\pi}\int_{-\infty}^{\infty} X(jw)e^{jwt}dt = \frac{1}{2\pi}\int_{-W}^{W} e^{jwt}dw = \frac{\sin Wt}{\pi t}$

令 $\text{sinc}(\theta)=\frac{\sin\pi\theta}{\pi\theta}$，则 $\frac{\sin Wt}{\pi t} = \frac{W}{\pi}\text{sinc}(\frac{Wt}{\pi})$

Fourier tranform for periodic signal

$x(t)$ 可用 FS 表示，也可用 FT 表示：

FS: $x(t)=\sum_{k=-\infty}^{\infty} a_k e^{jkw_0t}$

Ft: $x(t)=\frac{1}{2\pi}\int_{-\infty}^{\infty} X(jw)e^{jwt}dw$, $X(jw)=\int_{-\infty}^{+\infty} x(t)e^{jwt}dw$

对于 $x(t)=a_k e^{jkw_0t}$

a_k 与 $X(jw)$ 间有关系么？ $X(jw)=2\pi a_k\delta(w-kw_0)$ *

则 $X(jw)=\sum_{k=-\infty}^{\infty} a_k 2\pi\delta(w-kw_0)$

* 推导: $x(t)=\frac{1}{2\pi}\int_{-\infty}^{\infty} X(jw)e^{jwt}dw = a_k e^{jkw_0t}$

$\int_{-\infty}^{\infty} X(jw)e^{jwt}dw = 2\pi a_k e^{jkw_0t}$

应想: 仅 $w=kw_0$ 时，被积者为 0，故猜 $X(jw)$ 有 $\delta(w-kw_0)$

故 $w=kw_0$ 下: $?\,e^{jkw_0t}=2\pi a_k e^{jkw_0t}$, $?=2\pi a_k$

Properties of continuous-time FT

Notation: $x(t)=\frac{1}{2\pi}\int_{-\infty}^{\infty} X(jw)e^{jwt}dt$ $X(jw)=\int_{-\infty}^{\infty} x(t)e^{-jwt}dt$

$x(t) \overset{F}{\longleftrightarrow} X(jw)$, $X(jw)=F\{x(t)\}$

$x(t)=F^{-1}\{X(jw)\}$

补充：求 $x(t)=\sum_{k=-\infty}^{\infty}\delta(t-kT)$ 的FT，$T=T$.

(时域脉冲串)

$a_k=\frac{1}{T}\int_T \delta(t)e^{-jk\omega_0 t}dt=\frac{1}{T}$

$X(j\omega)=\sum_{k=-\infty}^{\infty}a_k 2\pi\delta(\omega-k\omega_0)=\frac{2\pi}{T}\sum_{k=-\infty}^{\infty}\delta(\omega-\frac{2k\pi}{T})$

线性：$x(t)\xleftrightarrow{F}X(j\omega)$ $\quad y(t)\xleftrightarrow{F}Y(j\omega)$

$ax(t)+by(t)\xleftrightarrow{F}aX(j\omega)+bY(j\omega)$.

时移：$x(t)\xleftrightarrow{F}X(j\omega)\Rightarrow x(t-t_0)\xleftrightarrow{F}e^{-j\omega t_0}X(j\omega)$

Proof: $x(t)=\frac{1}{2\pi}\int_{-\infty}^{+\infty}X(j\omega)e^{j\omega t}dt$

$x(t-t_0)=\frac{1}{2\pi}\int_{-\infty}^{\infty}X(j\omega)e^{j\omega(t-t_0)}d\omega=\frac{1}{2\pi}\int_{-\infty}^{\infty}(e^{-j\omega t_0}X(j\omega))e^{j\omega t}d\omega$

共轭：$x(t)\xleftrightarrow{F}X(j\omega)$ $\quad x^*(t)\xleftrightarrow{F}X^*(-j\omega)$.

Proof: $X^*(j\omega)=[\int_{-\infty}^{\infty}x(t)e^{-j\omega t}dt]^*=\int_{-\infty}^{\infty}x(t)^*e^{j\omega t}dt$.

$X^*(-j\omega)=\int_{-\infty}^{\infty}x(t)^*e^{-j\omega t}dt=F\{x^*(t)\}$.

△: Conjugation Symmetry: $X(-j\omega)=X^*(j\omega)$ （$x(t)$是实信号）

时逆：$x(t)\xleftrightarrow{F}X(j\omega)\Rightarrow x(-t)\xleftrightarrow{F}X(-j\omega)$

Diff & Int: $x(t)\xleftrightarrow{F}X(j\omega)\Rightarrow\frac{dx(t)}{dt}\xleftrightarrow{F}j\omega X(j\omega)$

$\int_{-\infty}^{t}x(\tau)d\tau\xleftrightarrow{F}\frac{1}{j\omega}X(j\omega)+\pi X(0)\delta(\omega)$

Proof: $\frac{dx(t)}{dt}=\frac{1}{2\pi}\int_{-\infty}^{\infty}X(j\omega)\frac{de^{j\omega t}}{dt}d\omega$

$=\frac{1}{2\pi}\int_{-\infty}^{\infty}X(j\omega)\cdot j\omega e^{j\omega t}d\omega$

$\int_{-\infty}^{t}x(\tau)d\tau=\int_{-\infty}^{t}\frac{1}{2\pi}\int_{-\infty}^{\infty}X(j\omega)e^{j\omega\tau}d\omega d\tau$

$=\frac{1}{2\pi}\int_{-\infty}^{\infty}X(j\omega)\int_{-\infty}^{t}e^{j\omega\tau}d\tau d\omega$

Date

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} X(jw)\left[\frac{e^{jwt}}{jw} - \lim_{\tau\to-\infty}\frac{e^{jw\tau}}{jw}\right]dw \quad ??$$

$$\int_{-\infty}^{t} x(\tau)d\tau = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(jw)\int_{-\infty}^{\infty} u(t-\tau)e^{jw\tau}d\tau dw$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} X(jw)\int_{-\infty}^{\infty} u(p)e^{jw(t-p)}dp\,dw$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} X(jw)\boxed{\int_{-\infty}^{\infty} e^{-jwp}dp}\cdot e^{jwt}dw$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} X(jw)\boxed{\left(\frac{1}{jw}+\pi\delta(w)\right)}e^{jwt}dw$$

*: $z(t) = \begin{cases} -e^{\alpha t}, & t<0 \\ e^{-\alpha t}, & t>0 \end{cases}$ $\alpha>0$, $sgn(t) = \lim_{\alpha\to 0} z(t)$

$$F\{z(t)\} = \int_{-\infty}^{0} -e^{\alpha t}e^{-jwt}dt + \int_{0}^{\infty} e^{-\alpha t}e^{-jwt}dt$$

$$= \frac{-2jw}{\alpha^2+w^2}$$

$$F\{sgn(t)\} = \lim_{\alpha\to 0} F\{z(t)\} = \lim_{\alpha\to 0}\frac{-2jw}{\alpha^2+w^2} = \frac{2}{jw}$$

$u(t) = \frac{1}{2}sgn(t) + \frac{1}{2}$, $F\{u(t)\} = \frac{1}{jw} + \pi\delta(w)$

奇偶 & 实: $x(t)$ real & even $\Rightarrow$ $X(jw)$ real & even

$x(t)$ real & odd $\Rightarrow$ $X(jw)$ purely imaginary & odd

$x(t)$ even $\Rightarrow$ $X(jw) = X(-jw)$

$x(t)$ real $\Rightarrow$ $X(-jw) = X^*(jw)$

$x(t) = x_e(t) + x_o(t)$

$F\{x(t)\} = F\{x_e(t)\} + F\{x_o(t)\}$ $\Leftrightarrow$ $Ev\{x(t)\} \xleftrightarrow{F} Re\{X(jw)\}$

$Od\{x(t)\} \xleftrightarrow{F} Im\{X(jw)\}$.