The Laplace Transform
(ch.9)

- The Laplace transform
- The region of convergence for Laplace transforms
- The inverse Laplace transform
- Geometric evaluation of the Fourier transform from the pole-zero plot
- Properties of the Laplace transform
- Some Laplace transform pairs
- Analysis and characterization of LTI systems using the Laplace transform
- System function algebra and block diagram representations
- The unilateral Laplace transform

The Laplace transform

Recall the response of LTI systems to complex exponentials

$$e^{st} \longrightarrow \boxed{h(t)} \longrightarrow y(t) = \int_{-\infty}^{\infty} h(\tau)e^{s(t-\tau)}d\tau = e^{st}\int_{-\infty}^{\infty} h(\tau)e^{-s\tau}d\tau$$

Define: $\int_{-\infty}^{\infty} h(\tau)e^{-s\tau}d\tau = H(s) \quad \rightarrow \quad y(t) = H(s)e^{st}$

Definition

$$\boxed{X(s) \triangleq \int_{-\infty}^{+\infty} x(t)e^{-st}dt}$$

$$x(t) \overset{\mathfrak{L}}{\longleftrightarrow} X(s)$$

The Laplace transform

Laplace transform vs Fourier transform

$$x(t) \overset{\mathfrak{L}}{\longleftrightarrow} X(s)$$

$$X(s) \triangleq \int_{-\infty}^{+\infty} x(t)e^{-st}dt$$

$s = j\omega \Downarrow$ $\qquad\qquad$ $\Downarrow s = \sigma + j\omega$

$$X(j\omega) = \int_{-\infty}^{+\infty} x(t)e^{-j\omega t}dt$$

$$X(s)\Big|_{s=j\omega} = \mathcal{F}\{x(t)\}$$

$$X(\sigma + j\omega) = \int_{-\infty}^{+\infty} x(t)e^{-(\sigma+j\omega)t}dt$$

$$X(\sigma + j\omega) = \int_{-\infty}^{+\infty} [x(t)e^{-\sigma t}]e^{-j\omega t}dt$$

$$X(s)\Big|_{s=\sigma+j\omega} = \mathcal{F}\{x(t)e^{-\sigma t}\}$$

$$\boxed{\mathfrak{L}\{x(t)\} \triangleq \mathcal{F}\{x(t)e^{-\sigma t}\}}$$

The Laplace transform

Examples

$x(t) = e^{-at}u(t) \qquad X(s) = ?$

Solution

FT: $$X(j\omega) = \int_0^{+\infty} e^{-at} e^{-j\omega t} dt = \int_0^{\infty} e^{-(j\omega + a)t} dt = \frac{1}{a + j\omega}, \qquad a > 0$$

LT: $$X(s) = \int_0^{\infty} e^{-at} e^{-st} dt = \int_0^{\infty} e^{-(\sigma + a)t} e^{-j\omega t} dt = \frac{1}{(\sigma + a) + j\omega}, \sigma + a > 0$$

$$s = \sigma + j\omega$$

$$\boxed{e^{-at}u(t) \xleftrightarrow{\mathfrak{L}} \frac{1}{s + a} \qquad \mathcal{Re}\{s\} > -a}$$

The Laplace transform

Examples

$$x(t) = -e^{-at}u(-t) \qquad\qquad X(s) = ?$$

Solution

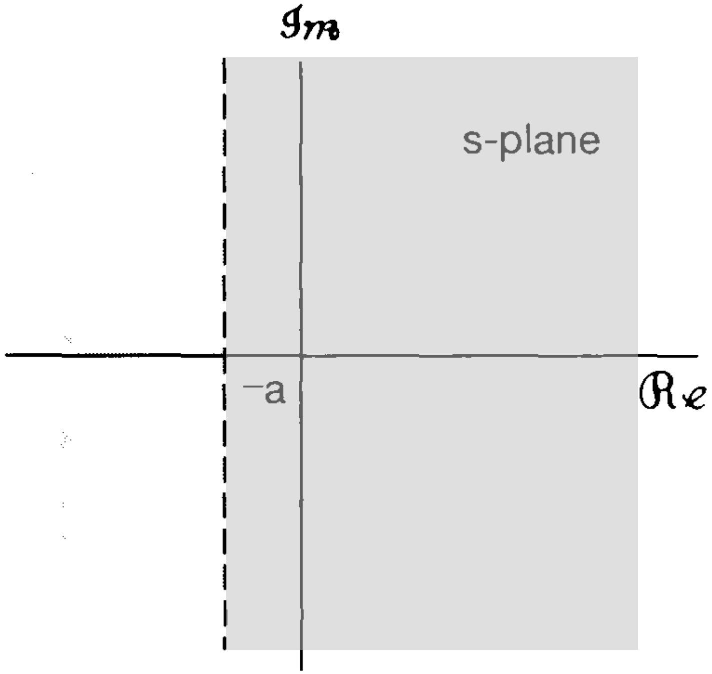
$$X(s) = -\int_{-\infty}^{+\infty} e^{-at}u(-t)e^{-st}dt = -\int_{-\infty}^{0} e^{-(s+a)t}dt = \frac{1}{s+a}, \qquad \mathcal{Re}\{s\} < -a$$

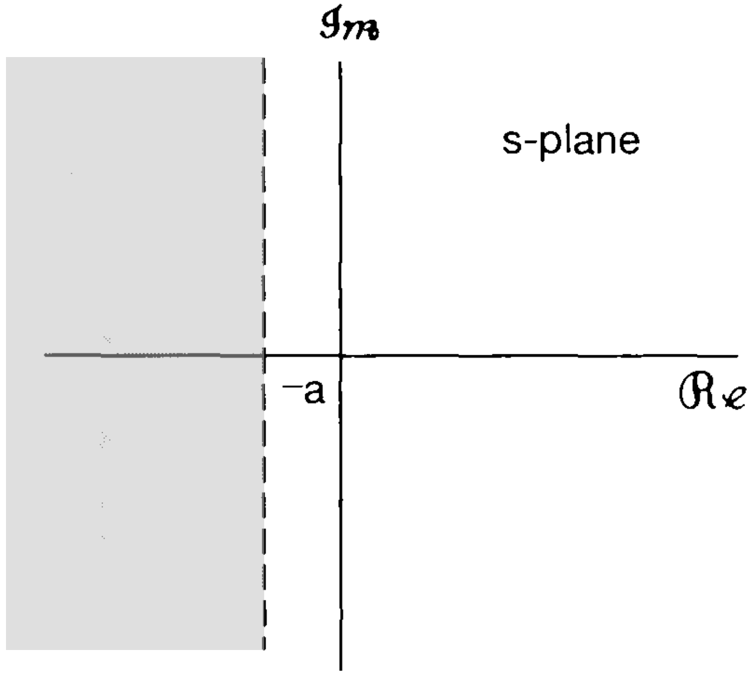
$$\boxed{-e^{-at}u(-t) \overset{\mathfrak{L}}{\longleftrightarrow} \frac{1}{s+a} \qquad \mathcal{Re}\{s\} < -a}$$

Region of convergence (ROC)

$$e^{-at}u(t) \overset{\mathfrak{L}}{\longleftrightarrow} \frac{1}{s+a} \quad \mathcal{R}e\{s\} > -a$$

$$-e^{-at}u(-t) \overset{\mathfrak{L}}{\longleftrightarrow} \frac{1}{s+a} \quad \mathcal{R}e\{s\} < -a$$





The Laplace transform

Examples

$$x(t) = 3e^{-2t}u(t) - 2e^{-t}u(t) \qquad\qquad X(s) = ?$$

Solution

$$X(s) = \int_{-\infty}^{+\infty} [3e^{-2t}u(t) - 2e^{-t}u(t)]e^{-st}dt$$

$$= 3\int_{-\infty}^{+\infty} e^{-2t}e^{-st}u(t)dt - 2\int_{-\infty}^{+\infty} e^{-t}e^{-st}u(t)dt = \frac{3}{s+2} - \frac{2}{s+1} = \frac{s-1}{s^2+3s+2}$$

$$e^{-t}u(t) \overset{\mathfrak{L}}{\longleftrightarrow} \frac{1}{s+1} \qquad \mathcal{Re}\{s\} > -1$$

$$e^{-2t}u(t) \overset{\mathfrak{L}}{\longleftrightarrow} \frac{1}{s+2} \qquad \mathcal{Re}\{s\} > -2$$

$$3e^{-2t}u(t) - 2e^{-2t}u(t) \overset{\mathfrak{L}}{\longleftrightarrow} \frac{s-1}{s^2+3s+2} \qquad \mathcal{Re}\{s\} > -1$$

The Laplace transform

Examples $x(t)=e^{-2t}u(t)+e^{-t}(\cos 3t)u(t) \qquad X(s)=?$

Solution

$$x(t)=\left[e^{-2t}+\frac{1}{2}e^{-(1-3j)t}+\frac{1}{2}e^{-(1+3j)t}\right]u(t)$$

$$X(s)=\int_{-\infty}^{+\infty}e^{-2t}u(t)e^{-st}dt+\frac{1}{2}\int_{-\infty}^{+\infty}e^{-(1-3j)t}u(t)e^{-st}dt+\frac{1}{2}\int_{-\infty}^{+\infty}e^{-(1+3j)t}u(t)e^{-st}dt$$

$$e^{-2t}u(t)\overset{\mathfrak{L}}{\longleftrightarrow}\frac{1}{s+2} \qquad \mathcal{R}e\{s\}>-2$$

$$e^{-(1-3j)t}u(t)\overset{\mathfrak{L}}{\longleftrightarrow}\frac{1}{s+(1-3j)} \qquad \mathcal{R}e\{s\}>-1$$

$$e^{-(1+3j)t}u(t)\overset{\mathfrak{L}}{\longleftrightarrow}\frac{1}{s+(1+3j)} \qquad \mathcal{R}e\{s\}>-1$$

$$X(s)=\frac{1}{s+2}+\frac{1}{2}\left(\frac{1}{s+(1-3j)}\right)+\frac{1}{2}\left(\frac{1}{s+(1+3j)}\right)=\frac{2s^2+5s+12}{(s^2+2s+10)(s+2)}, \mathcal{R}e\{s\}>-1$$

Pole-zero plot of rational $X(s)$

$$X(s) = \frac{N(s)}{D(S)}$$

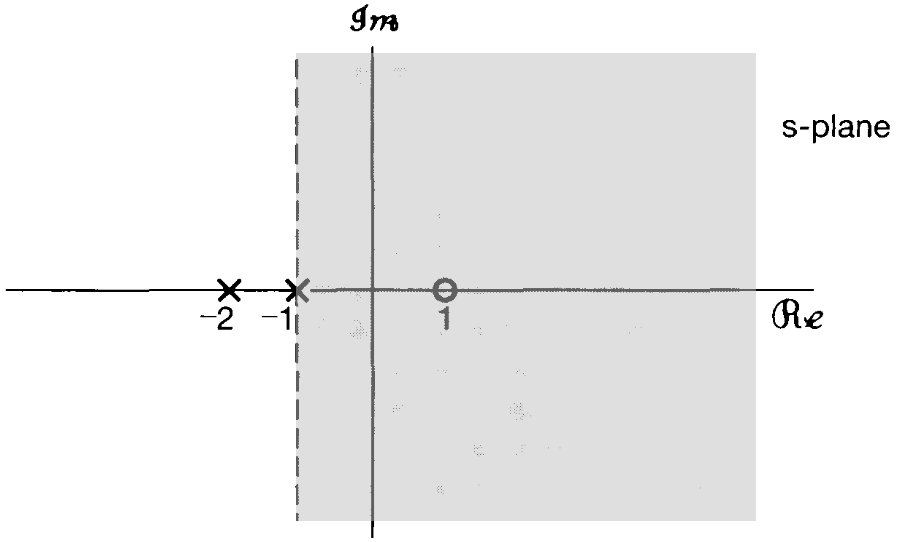
"x": the location of the root of the numerator polynomial

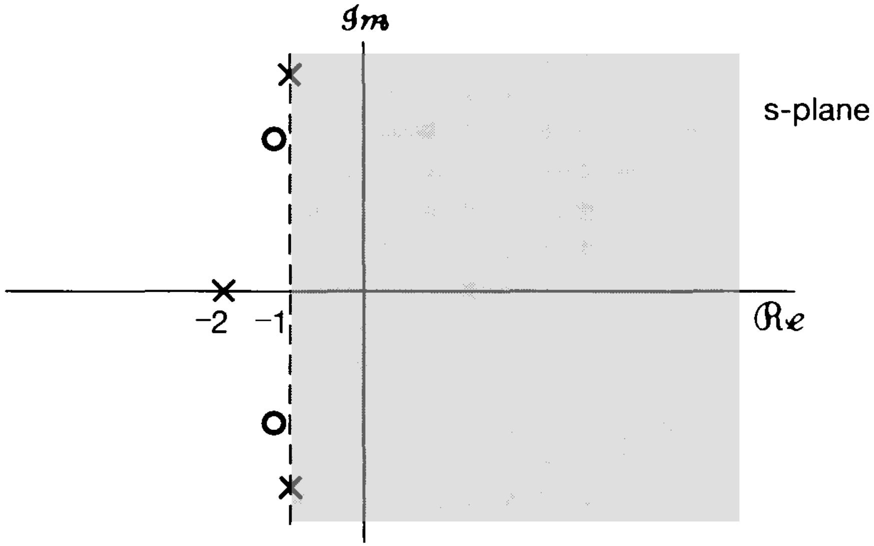
"o": the location of the root of the denominator polynomial

Examples

$$X(s) = \frac{s-1}{s^2+3s+2}, \mathcal{Re}\{s\} > -1$$

$$X(s) = \frac{2s^2+5s+12}{(s^2+2s+10)(s+2)}, \mathcal{Re}\{s\} > -1$$





The Laplace transform

Examples

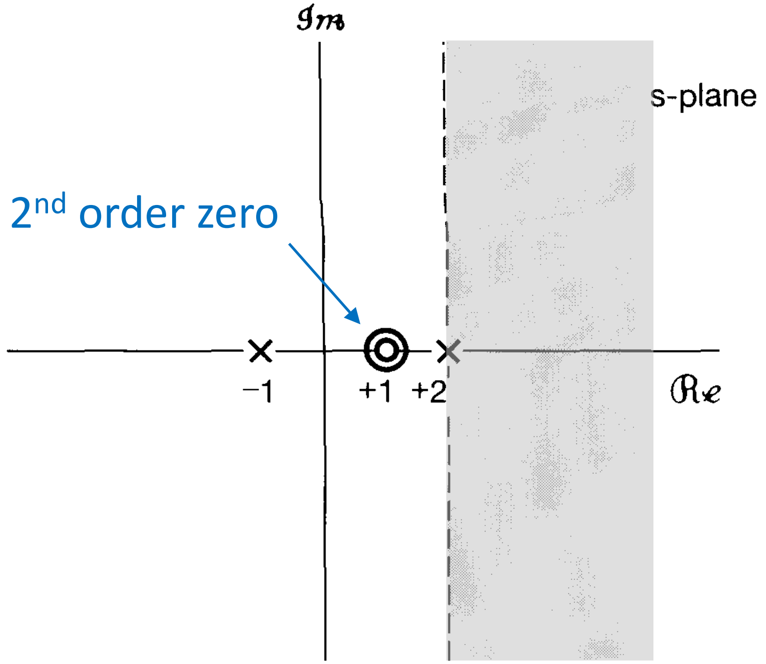
$$x(t) = \delta(t) - \frac{4}{3}e^{-t}u(t) + \frac{1}{3}e^{2t}u(t) \qquad X(s) = ?$$

Solution

$$\mathfrak{L}\{\delta(t)\} = \int_{-\infty}^{+\infty} \delta(t)e^{-st}dt = 1$$ valid for any value of s

$$X(s) = 1 - \frac{4}{3}\frac{1}{s+1} + \frac{1}{3}\frac{1}{s-2} \qquad \mathcal{Re}\{s\} > 2$$

$$= \frac{(s-1)^2}{(s+1)(s-2)} \qquad \mathcal{Re}\{s\} > 2$$



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The region of convergence for Laplace transforms

Properties

1. The ROC of $X(s)$ consists of strips parallel to the $j\omega$-axis in the s-plane

ROC of $X(s)$: Fourier transform of $x(t)e^{-\sigma t}$ converges (absolutely integrable)

$$\int_{-\infty}^{+\infty} |x(t)|e^{-\sigma t}dt < \infty$$

depends only on σ, the real part of s

2. For rational Laplace transforms, the ROC does not contain any poles.

$X(s)$ is infinite at a pole

The region of convergence for Laplace transforms

Properties

3. If $x(t)$ is of finite duration and is absolutely integrable, then the ROC is the entire s-plane.

For convergence, require

$$\int_{T_1}^{T_2} |x(t)| e^{-\sigma t} dt < \infty$$

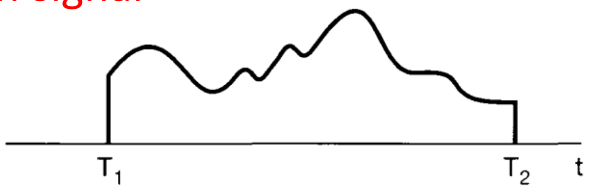
If $\sigma > 0$,

$$\int_{T_1}^{T_2} |x(t)| e^{-\sigma t} dt \le e^{-\sigma T_1} \int_{T_1}^{T_2} |x(t)| dt$$

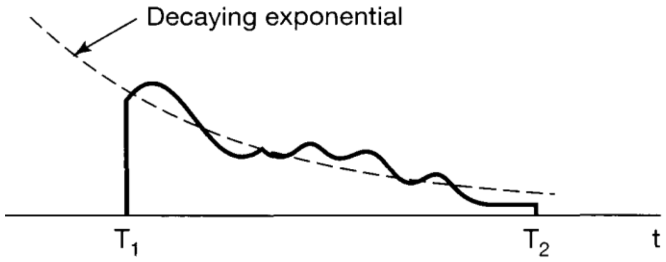
If $\sigma < 0$,

$$\int_{T_1}^{T_2} |x(t)| e^{-\sigma t} dt \le e^{-\sigma T_2} \int_{T_1}^{T_2} |x(t)| dt$$

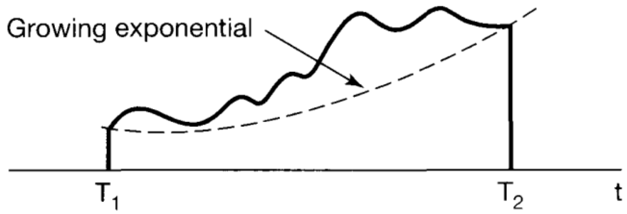
Finite-duration signal



Multiplied by a decaying exponential



Multiplied by a growing exponential



Examples

$$x(t) = \begin{cases} e^{-at} & 0 < t < T \\ 0 & \text{otherwise} \end{cases} \qquad X(s) = ?$$

Solution

$$X(s) = \int_0^T e^{-at} e^{-st} dt = \frac{1}{s+a}\left[1 - e^{-(s+a)T}\right]$$

$$\lim_{s \to -a} X(s) = \lim_{s \to -a} \left[\frac{\frac{d}{ds}\left(1 - e^{-(s+a)T}\right)}{\frac{d}{ds}(s+a)} \right] = \lim_{s \to -a} T e^{-aT} e^{-sT}$$

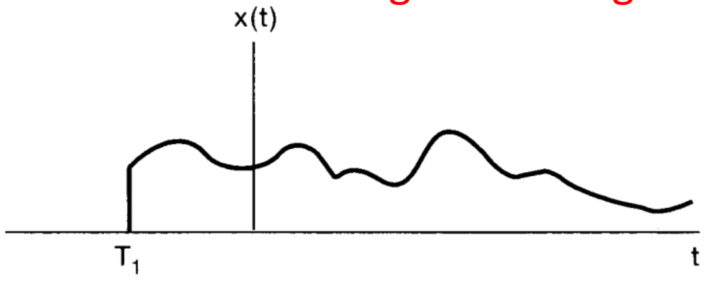
$$X(-a) = T$$

ROC = the entire s-plane

Properties

4. If $x(t)$ is right-sided, and if the line $\mathcal{R}e\{s\} = \sigma_0$ is in the ROC, then all values of s for which $\mathcal{R}e\{s\} > \sigma_0$ will also be in the ROC.

Right-sided signal



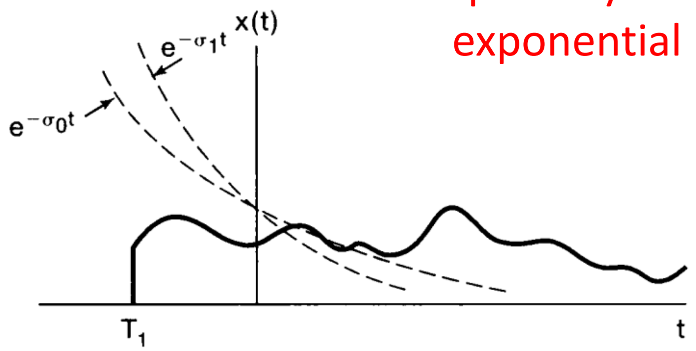
For convergence, require $\int_{T_1}^{+\infty} |x(t)| e^{-\sigma_0 t} dt < \infty$

For $\sigma_1 > \sigma_0$,

$$\int_{T_1}^{+\infty} |x(t)| e^{-\sigma_1 t} dt = \int_{T_1}^{+\infty} |x(t)| e^{-\sigma_0 t} e^{-(\sigma_1 - \sigma_0)t} dt$$

$$\leq e^{-(\sigma_1 - \sigma_0)T_1} \int_{T_1}^{+\infty} |x(t)| e^{-\sigma_0 t} dt$$

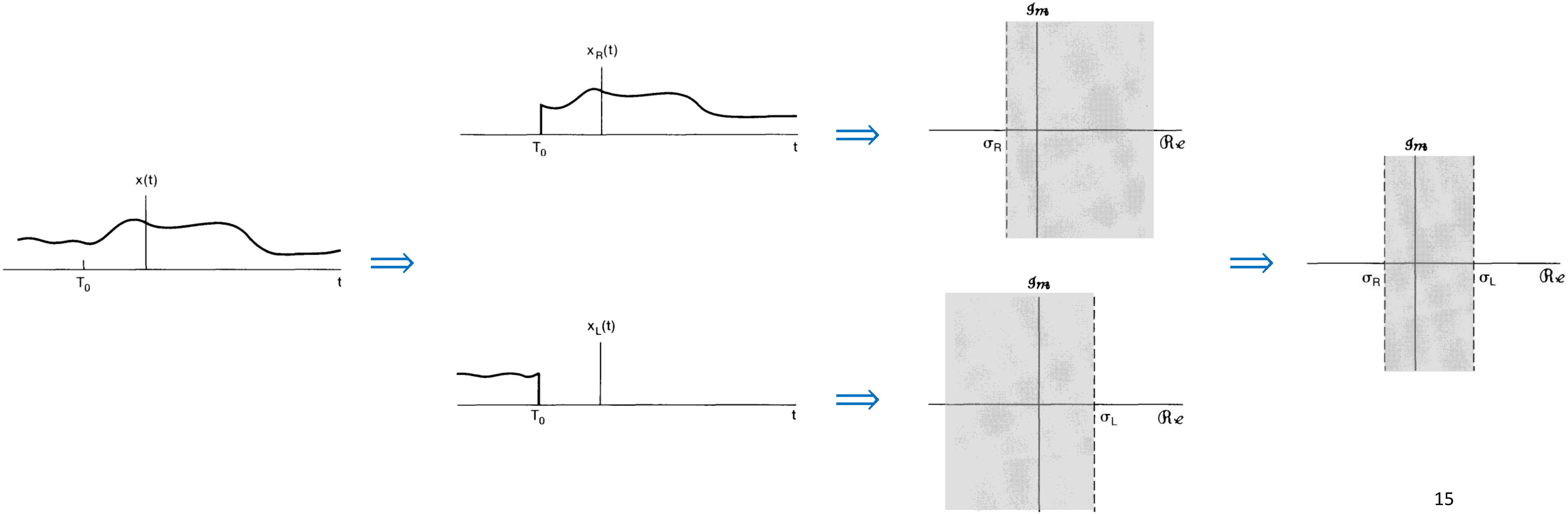
Multiplied by an exponential



5. If $x(t)$ is left-sided, and if the line $\mathcal{R}e\{s\} = \sigma_0$ is in the ROC, then all values of s for which $\mathcal{R}e\{s\} < \sigma_0$ will also be in the ROC.

Properties

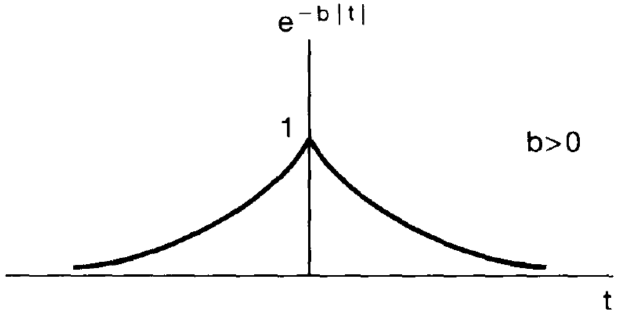
6. If $x(t)$ is two-sided, and if the line $\mathcal{R}e\{s\} = \sigma_0$ is in the ROC, then the ROC will consist of a strip in the s-plane that includes the line $\mathcal{R}e\{s\} = \sigma_0$.

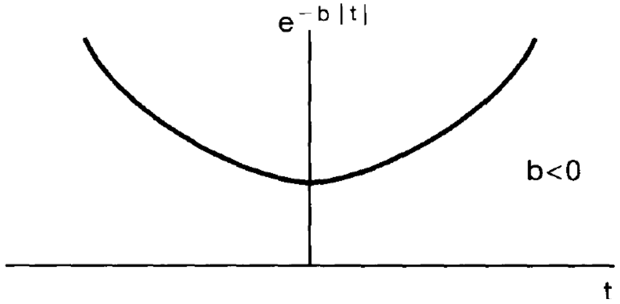


The region of convergence for Laplace transforms

Examples

$$x(t) = e^{-b|t|} \qquad X(s) = ?$$





Solution

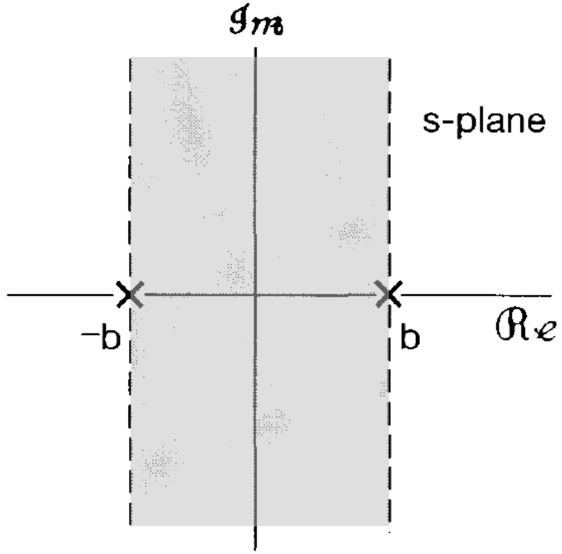
$$x(t) = e^{-bt}u(t) + e^{bt}u(-t)$$

$$e^{-bt}u(t) \overset{\mathfrak{L}}{\longleftrightarrow} \frac{1}{s+b} \qquad \mathcal{Re}\{s\} > -b$$

$$e^{bt}u(-t) \overset{\mathfrak{L}}{\longleftrightarrow} \frac{-1}{s-b} \qquad \mathcal{Re}\{s\} < b$$

$$e^{-b|t|} \overset{\mathfrak{L}}{\longleftrightarrow} \frac{1}{s+b} - \frac{1}{s-b} = -\frac{2b}{s^2-b^2} \qquad -b < \mathcal{Re}\{s\} < b$$

for $b > 0$



The region of convergence for Laplace transforms

Properties

7. If the Laplace transform $X(s)$ of $x(t)$ is rational, then its ROC is bounded by poles or extends to infinity. No poles are contained in the ROC.

- ❑ If $x(t)$ is left-sided, and if the line $\mathcal{Re}\{s\} = \sigma_0$ is in the ROC, then all values of s for which $\mathcal{Re}\{s\} < \sigma_0$ will also be in the ROC.
- ❑ If $x(t)$ is right-sided, and if the line $\mathcal{Re}\{s\} = \sigma_0$ is in the ROC, then all values of s for which $\mathcal{Re}\{s\} > \sigma_0$ will also be in the ROC.

8. If the Laplace transform $X(s)$ of $x(t)$ is rational, then if $x(t)$ is right-sided, the ROC is the region in the s-plane to the right of the right-most pole. The same applies to the left.

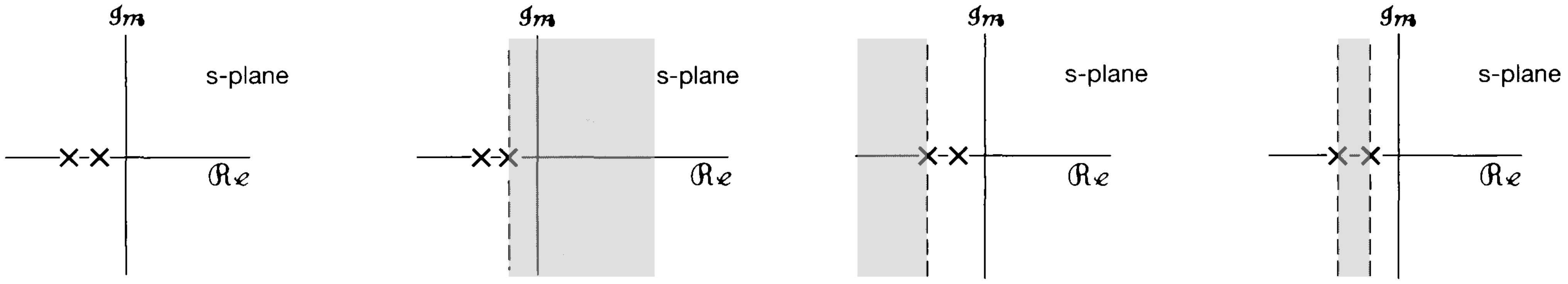
The region of convergence for Laplace transforms

Examples

$$X(s) = \frac{1}{(s+1)(s+2)}$$

ROCs and convergence of FT?

Solution



Right-sided	Left-sided	Two-sided
FT converges	Has no FT	Has no FT

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The inverse Laplace transform

$$X(\sigma + j\omega) = \mathcal{F}\{x(t)e^{-\sigma t}\} = \int_{-\infty}^{+\infty} [x(t)e^{-\sigma t}]e^{-j\omega t}dt$$

$$x(t)e^{-\sigma t} = \mathcal{F}^{-1}\{X(\sigma + j\omega)\} = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(\sigma + j\omega)e^{j\omega t}d\omega$$

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(\sigma + j\omega)e^{(\sigma + j\omega)t}d\omega$$

$s = \sigma + j\omega$
$ds = jd\omega$

$$\boxed{x(t) = \frac{1}{2\pi j}\int_{\sigma - j\infty}^{\sigma + j\infty} X(s)e^{st}ds}$$

The inverse Laplace transform

Examples

$$X(s) = \frac{1}{(s+1)(s+2)}, \qquad \mathcal{Re}\{s\} > -1 \qquad x(t) = ?$$

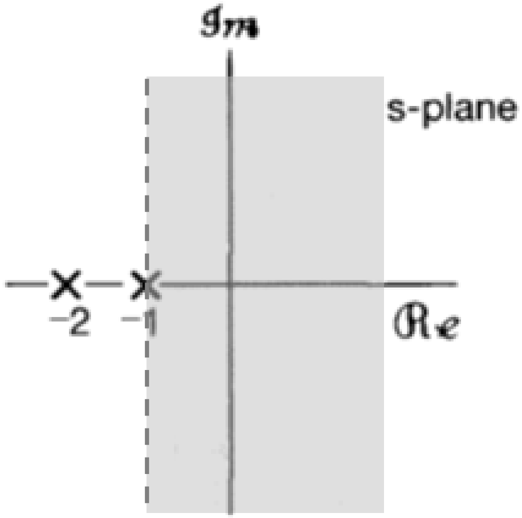
Solution

$$X(s) = \frac{1}{(s+1)(s+2)} = \frac{1}{s+1} - \frac{1}{s+2}$$

$$e^{-t}u(t) \overset{\mathfrak{L}}{\longleftrightarrow} \frac{1}{s+1} \qquad \mathcal{Re}\{s\} > -1$$

$$e^{-2t}u(t) \overset{\mathfrak{L}}{\longleftrightarrow} \frac{1}{s+2} \qquad \mathcal{Re}\{s\} > -2$$

$$x(t) = (e^{-t} - e^{-2t})u(t)$$



The inverse Laplace transform

Examples

$$X(s) = \frac{1}{(s+1)(s+2)}, \qquad \mathcal{Re}\{s\} < -2 \qquad x(t) = ?$$

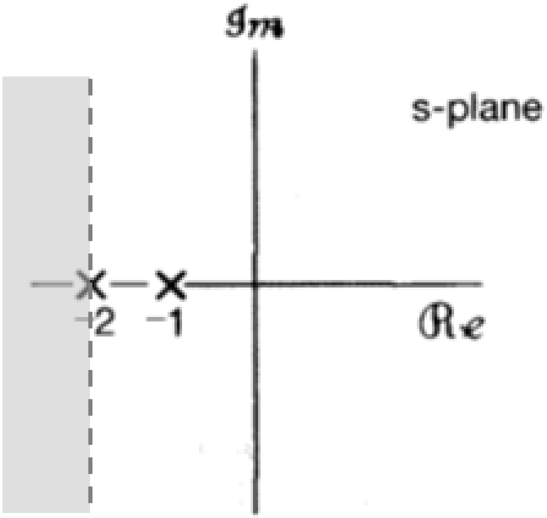
Solution

$$X(s) = \frac{1}{(s+1)(s+2)} = \frac{1}{s+1} - \frac{1}{s+2}$$

$$-e^{-t}u(-t) \overset{\mathfrak{L}}{\longleftrightarrow} \frac{1}{s+1} \qquad \mathcal{Re}\{s\} < -1$$

$$-e^{-2t}u(-t) \overset{\mathfrak{L}}{\longleftrightarrow} \frac{1}{s+2} \qquad \mathcal{Re}\{s\} < -2$$

$$x(t) = (-e^{-t} + e^{-2t})u(-t)$$



The inverse Laplace transform

Examples

$$X(s) = \frac{1}{(s+1)(s+2)}, \qquad -2 < \mathcal{Re}\{s\} < -1 \quad x(t) = ?$$

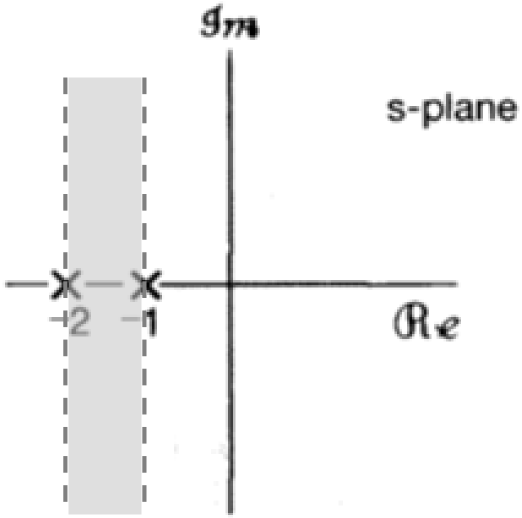
Solution

$$X(s) = \frac{1}{(s+1)(s+2)} = \frac{1}{s+1} - \frac{1}{s+2}$$

$$-e^{-t}u(-t) \overset{\mathfrak{L}}{\longleftrightarrow} \frac{1}{s+1} \qquad \mathcal{Re}\{s\} < -1$$

$$e^{-2t}u(t) \overset{\mathfrak{L}}{\longleftrightarrow} \frac{1}{s+2} \qquad \mathcal{Re}\{s\} > -2$$

$$x(t) = -e^{-t}u(-t) - e^{-2t}u(t)$$



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Geometry evaluation of the Fourier transform from the pole-zero plot

- Consider $X(s) = s - a$

$$|X(s_1)| = |\overrightarrow{s_1 - a}|$$

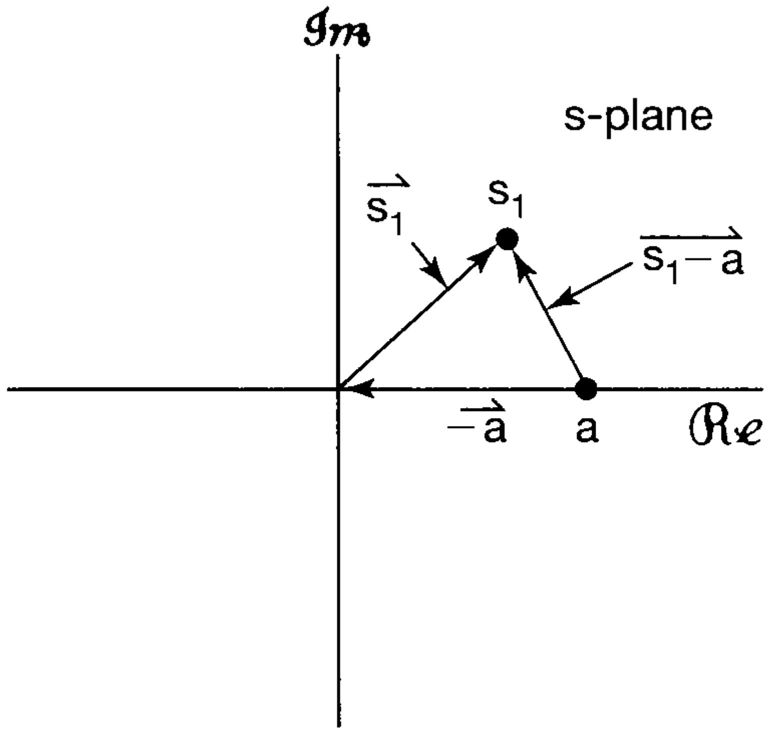
$$\sphericalangle X(s_1) = \sphericalangle \overrightarrow{s_1 - a}$$

- Consider $X(s) = 1/(s - a)$

$$|X(s_1)| = \frac{1}{|\overrightarrow{s_1 - a}|}$$

$$\sphericalangle X(s_1) = -\sphericalangle \overrightarrow{s_1 - a}$$

- Consider $X(s) = M \frac{\prod_{i=1}^{R}(s - \beta_i)}{\prod_{j=1}^{P}(s - \alpha_j)}$



$$|X(s_1)| = |M| \frac{\prod_{i=1}^{R}|s_1 - \beta_i|}{\prod_{j=1}^{P}|s_1 - \alpha_j|} \qquad \sphericalangle X(s_1) = \sphericalangle M + \sum_{i=1}^{R} \sphericalangle \overrightarrow{s_1 - \beta_i} - \sum_{i=1}^{P} \sphericalangle \overrightarrow{s_1 - \alpha_j}$$

Geometry evaluation of the Fourier transform from the pole-zero plot

Examples

$$X(s) = \frac{1}{s + 1/2}, \qquad \mathcal{Re}\{s\} > -\frac{1}{2}$$

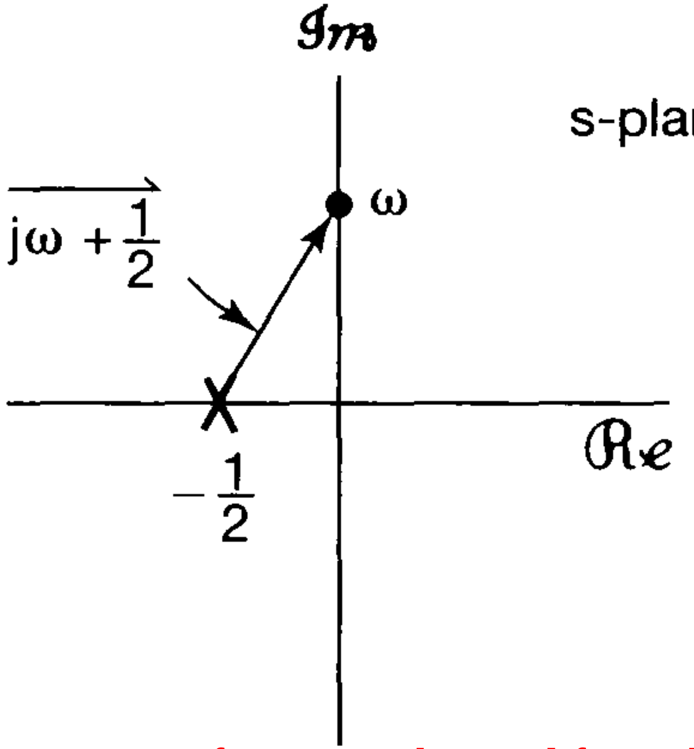
Magnitude and angle at $s = j\omega$?

Solution

$$X(j\omega) = \frac{1}{j\omega + 1/2}$$

$$|X(j\omega)|^2 = \frac{1}{\omega^2 + (1/2)^2}$$

$$\sphericalangle X(j\omega) = -\tan^{-1} 2\omega$$



Behavior of the Fourier transform can obtained from the pole-zero plot

Geometry evaluation of the Fourier transform from the pole-zero plot

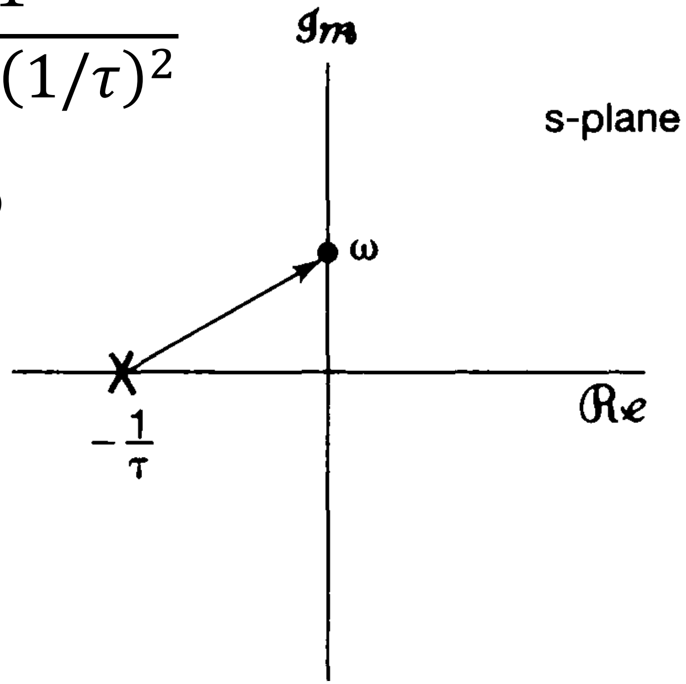
First-order systems

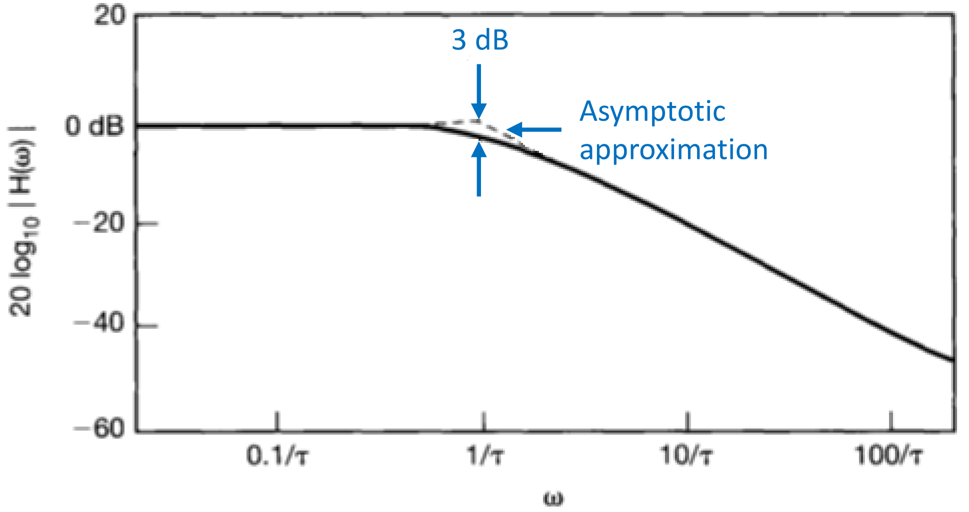
Consider $h(t) = \frac{1}{\tau} e^{-\frac{t}{\tau}} u(t)$

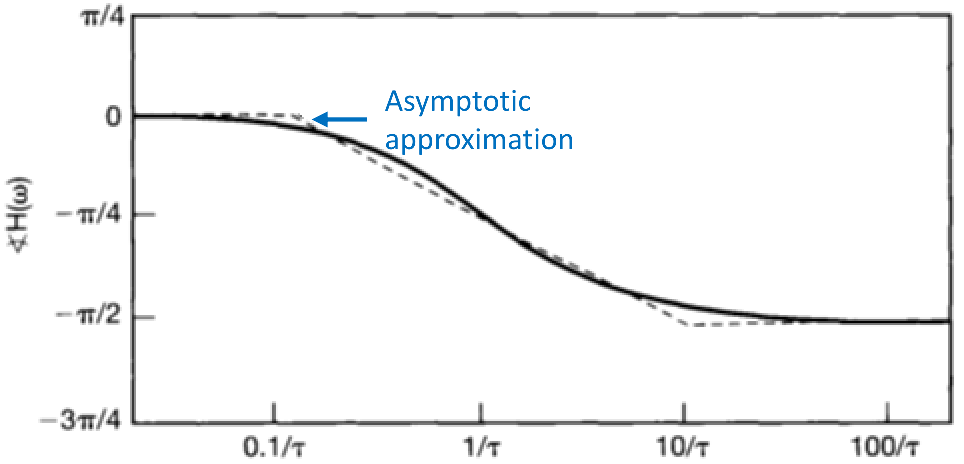
$$H(s) = \frac{1}{s\tau + 1}, \qquad \mathcal{Re}\{s\} > -\frac{1}{\tau}$$

$$|H(j\omega)|^2 = \frac{1}{\tau^2} \cdot \frac{1}{\omega^2 + (1/\tau)^2}$$

$$\sphericalangle H(j\omega) = -\tan^{-1} \tau\omega$$







Geometry evaluation of the Fourier transform from the pole-zero plot

Second-order systems

$$h(t) = M[e^{c_1 t} - e^{c_2 t}]u(t)$$

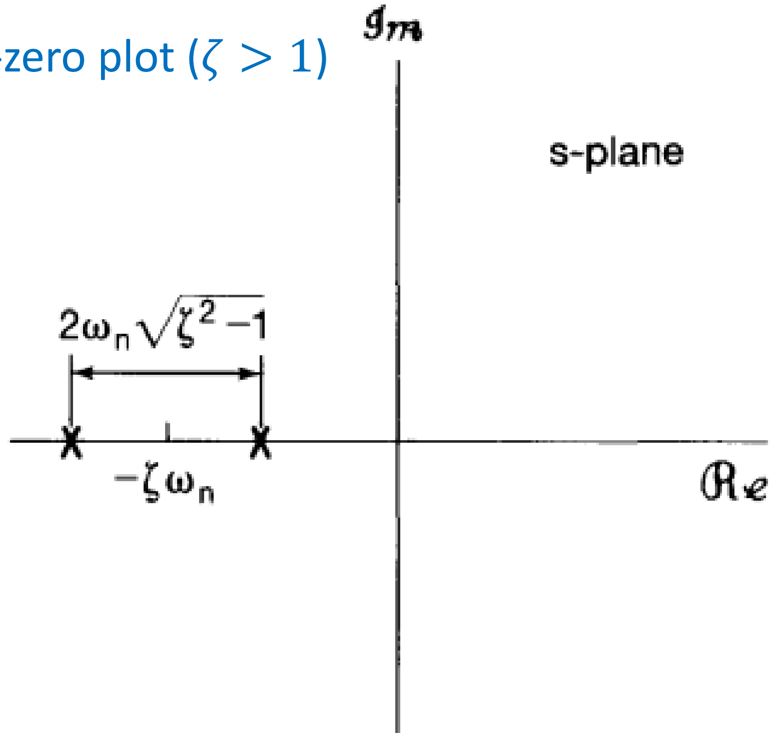
$$H(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} = \frac{\omega_n^2}{(s - c_1)(s - c_2)}$$

$$c_1 = -\zeta\omega_n + \omega_n\sqrt{\zeta^2 - 1}$$

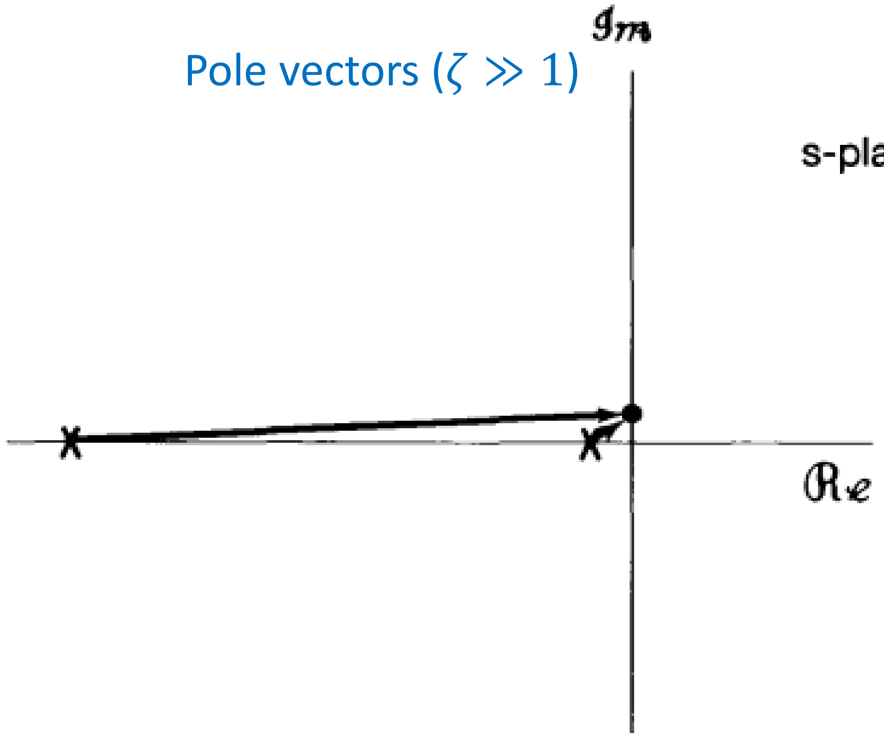
$$c_2 = -\zeta\omega_n - \omega_n\sqrt{\zeta^2 - 1}$$

$$M = \frac{\omega_n}{2\sqrt{\zeta^2 - 1}}$$

Pole-zero plot ($\zeta > 1$)



Pole vectors ($\zeta \gg 1$)



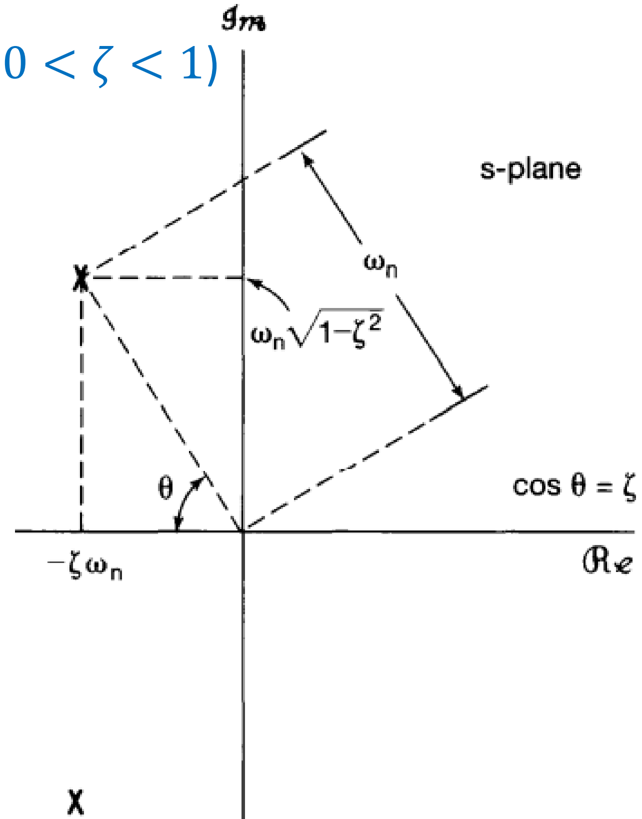
Second-order systems

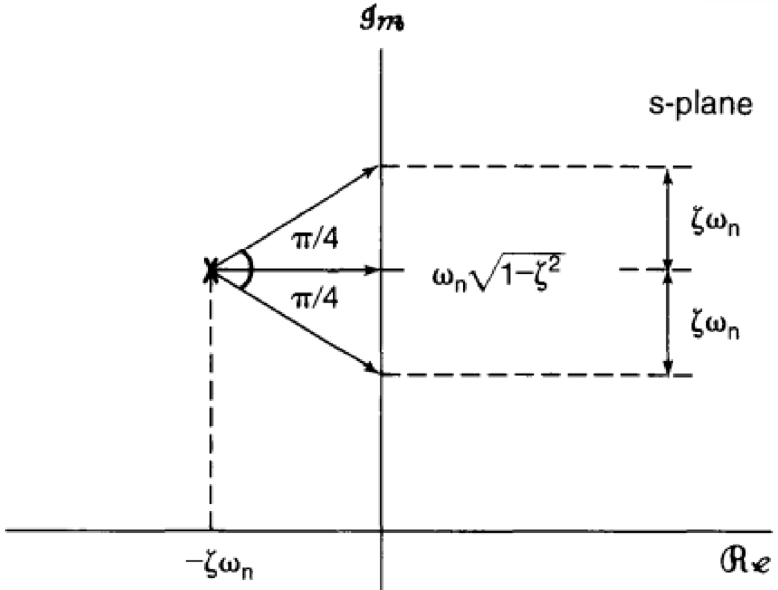
$$h(t) = M[e^{c_1 t} - e^{c_2 t}]u(t)$$

$$H(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} = \frac{\omega_n^2}{(s-c_1)(s-c_2)}$$

$$c_1 = -\zeta\omega_n + \omega_n\sqrt{\zeta^2-1}$$
$$c_2 = -\zeta\omega_n - \omega_n\sqrt{\zeta^2-1}$$
$$M = \frac{\omega_n}{2\sqrt{\zeta^2-1}}$$

Pole-zero plot ($0 < \zeta < 1$)





Pole vectors ($0 < \zeta < 1$)

$\omega = \omega_n\sqrt{\zeta^2-1}$

and $\omega = \omega_n\sqrt{\zeta^2-1} \pm \zeta\omega_n)$

Geometry evaluation of the Fourier transform from the pole-zero plot

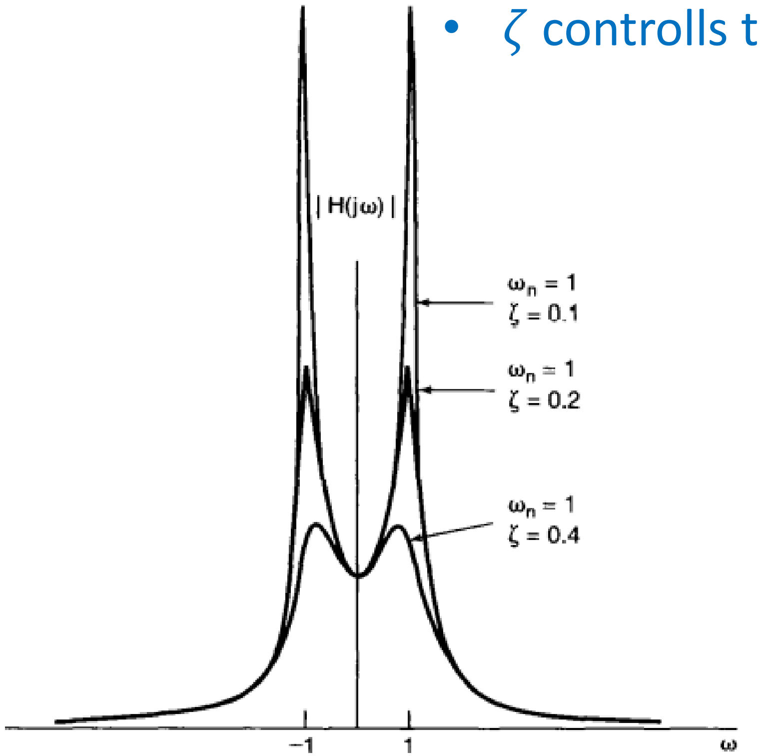
Second-order systems

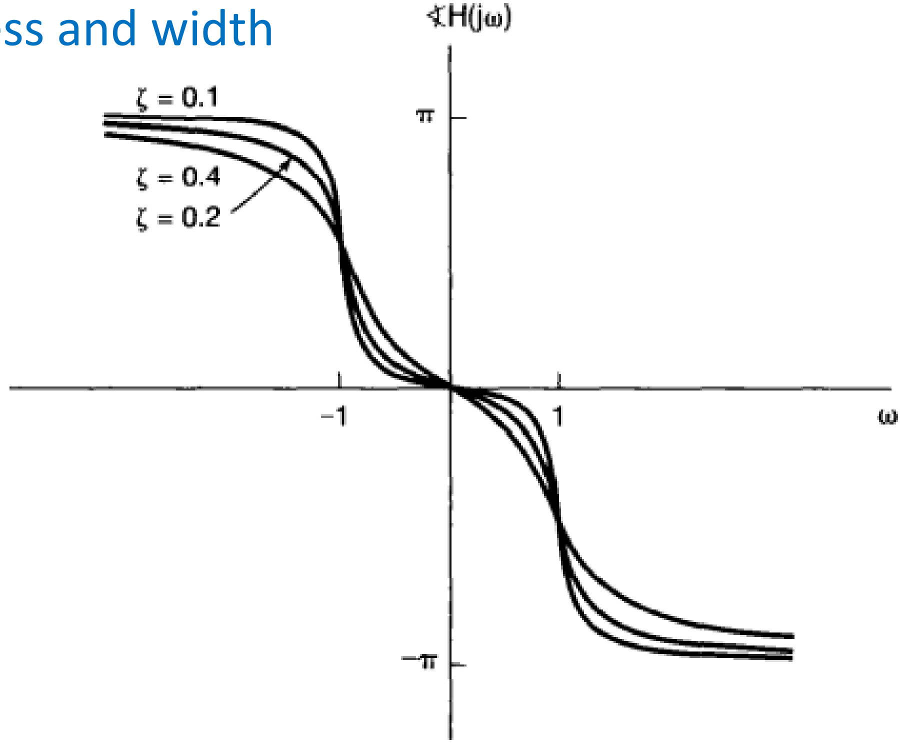
$$h(t) = M[e^{c_1 t} - e^{c_2 t}]u(t)$$

$$H(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} = \frac{\omega_n^2}{(s-c_1)(s-c_2)}$$

$$c_1 = -\zeta\omega_n + \omega_n\sqrt{\zeta^2 - 1}$$
$$c_2 = -\zeta\omega_n - \omega_n\sqrt{\zeta^2 - 1}$$
$$M = \frac{\omega_n}{2\sqrt{\zeta^2 - 1}}$$

- a non-ideal band-pass filter
- ζ controlls the sharpness and width

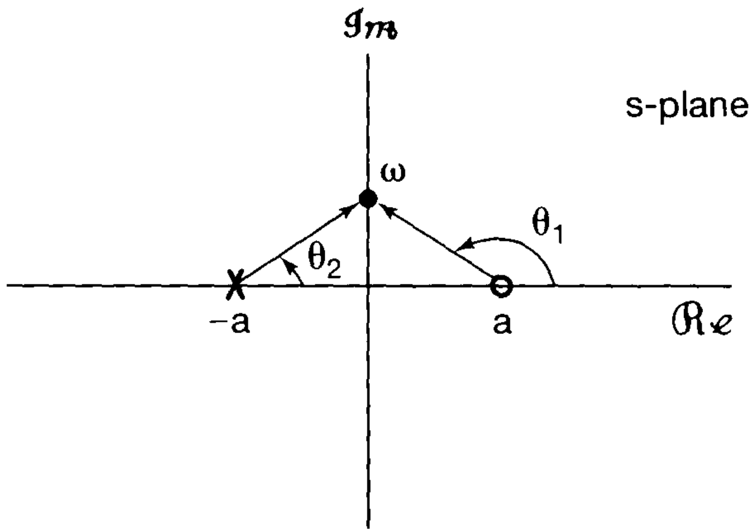


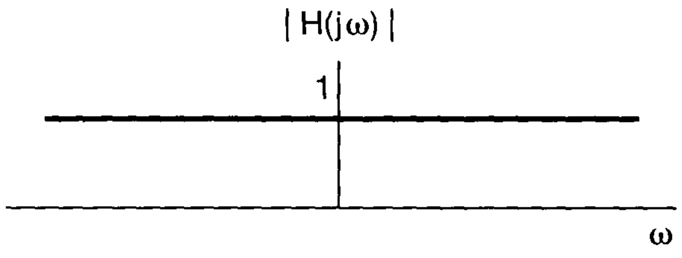


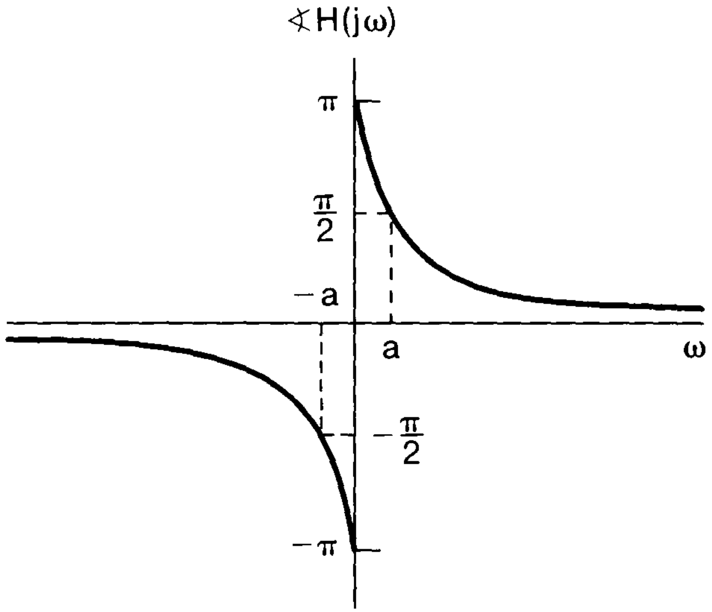
Geometry evaluation of the Fourier transform from the pole-zero plot

All-pass systems

$$\sphericalangle H(j\omega) = \theta_1 - \theta_2 = \pi - 2\theta_2 = \pi - 2\tan^{-1}\left(\frac{\omega}{a}\right)$$







The Laplace Transform
(ch.9)

- The Laplace transform
- The region of convergence for Laplace transforms
- The inverse Laplace transform
- Geometric evaluation of the Fourier transform from the pole-zero plot
- Properties of the Laplace transform
- Some Laplace transform pairs
- Analysis and characterization of LTI systems using the Laplace transform
- System function algebra and block diagram representations
- The unilateral Laplace transform

Properties of the Laplace transform

Linearity

$$x_1(t) \xleftrightarrow{\mathfrak{L}} X_1(s) \quad \text{ROC} = R1$$

$$\Longrightarrow x(t) = ax_1(t)+bx_2(t) \xleftrightarrow{\mathfrak{L}} aX_1(s) + bX_2(s)$$

$$x_2(t) \xleftrightarrow{\mathfrak{L}} X_2(s) \quad \text{ROC} = R2$$

ROC contains $R1 \cap R2$

$R1 \cap R2$ is can be empty: $x(t)$ has no Laplace transform

ROC of $X(s)$ can also be larger than $R1 \cap R2$

Properties of the Laplace transform

Example

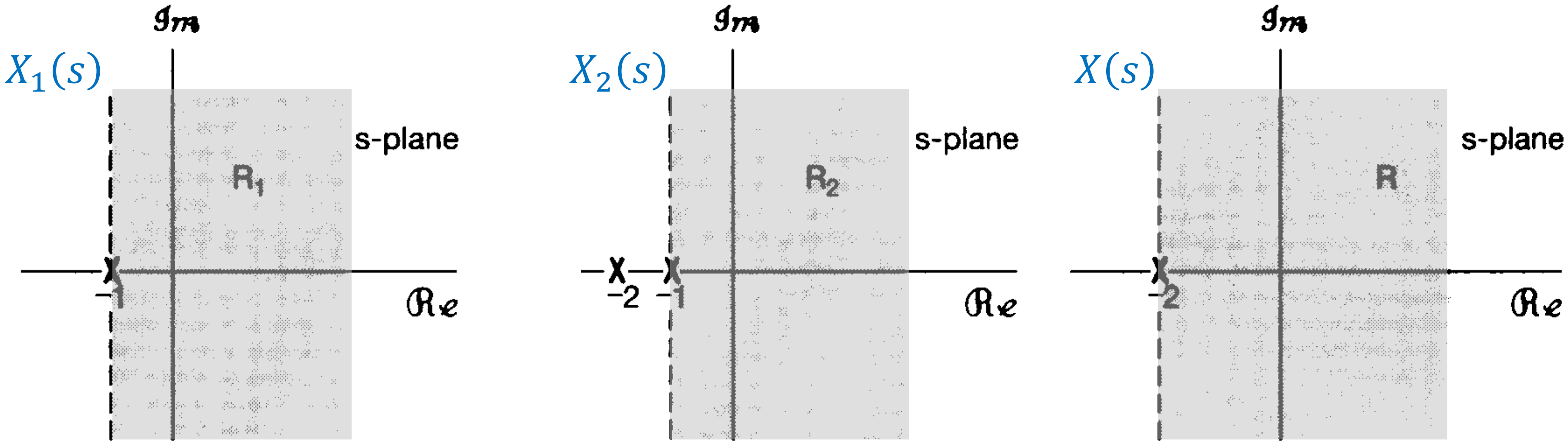
Consider $x(t) = x_1(t) - x_2(t)$

$$X_1(s) = \frac{1}{s+1}, \mathcal{Re}\{s\} > -1 \qquad X_2(s) = \frac{1}{(s+1)(s+2)}, \mathcal{Re}\{s\} > -1$$

$X(s) = ?$

Solution

$$X(s) = \frac{1}{s+1} - \frac{1}{(s+1)(s+2)} = \frac{s+1}{(s+1)(s+2)} = \frac{1}{(s+2)}, \qquad \mathcal{Re}\{s\} > -2$$



Properties of the Laplace transform

Time shifting

$$x(t) \overset{\mathfrak{L}}{\longleftrightarrow} X(s) \qquad \text{ROC} = R$$

$$\Downarrow$$

$$x(t - t_0) \overset{\mathfrak{L}}{\longleftrightarrow} e^{-st_0}X(s) \qquad \text{ROC} = R$$

Shifting in the s-domain

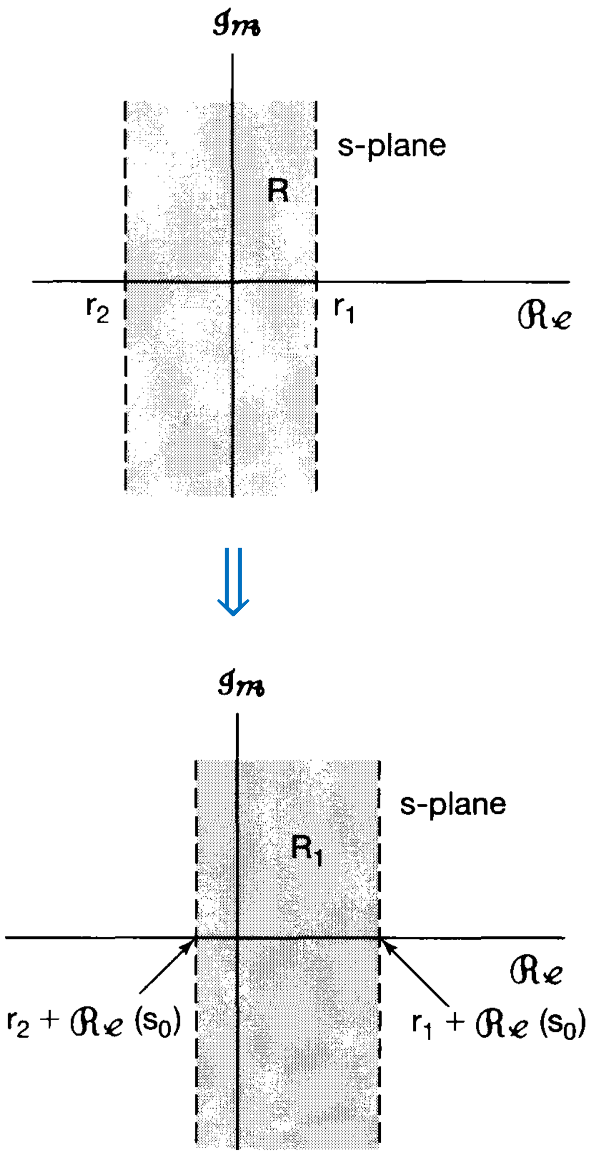
$$x(t) \overset{\mathfrak{L}}{\longleftrightarrow} X(s) \quad \text{ROC} = R$$

$$\Downarrow$$

$$e^{s_0 t}x(t) \overset{\mathfrak{L}}{\longleftrightarrow} X(s - s_0) \quad \text{ROC} = R + \mathcal{Re}\{s_0\}$$

$$\Downarrow \; s_0 = j\omega_0$$

$$e^{j\omega_0 t}x(t) \overset{\mathfrak{L}}{\longleftrightarrow} X(s - j\omega_0) \quad \text{ROC} = R$$



Properties of the Laplace transform

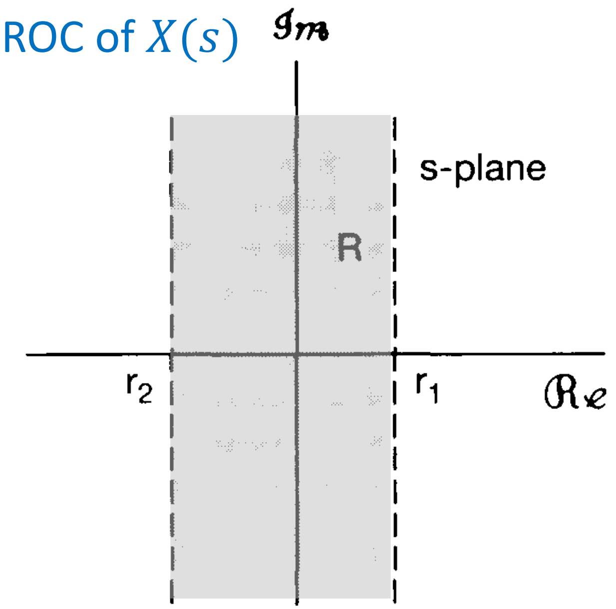
Time scaling

$$x(t) \overset{\mathfrak{L}}{\longleftrightarrow} X(s) \qquad \text{ROC} = R$$

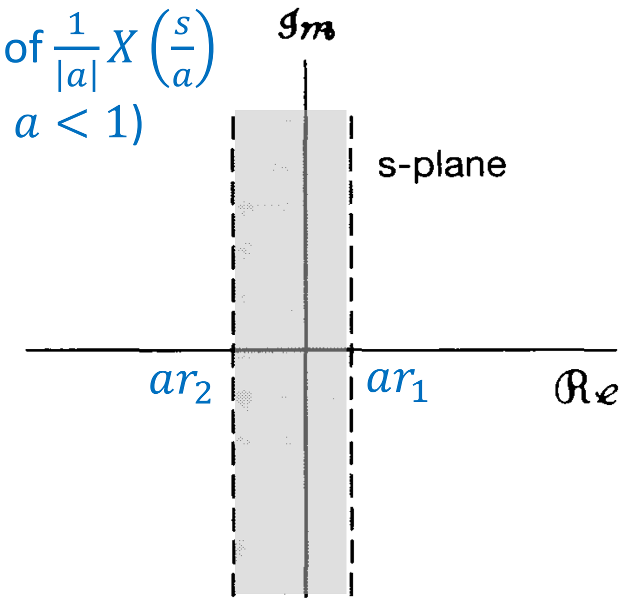
$$\Downarrow$$

$$x(at) \overset{\mathfrak{L}}{\longleftrightarrow} \frac{1}{|a|} X\left(\frac{s}{a}\right) \qquad \text{ROC} = aR \qquad \overset{a=-1}{\Longrightarrow} \qquad x(-t) \overset{\mathfrak{L}}{\longleftrightarrow} X(-s) \qquad \text{ROC} = -R$$

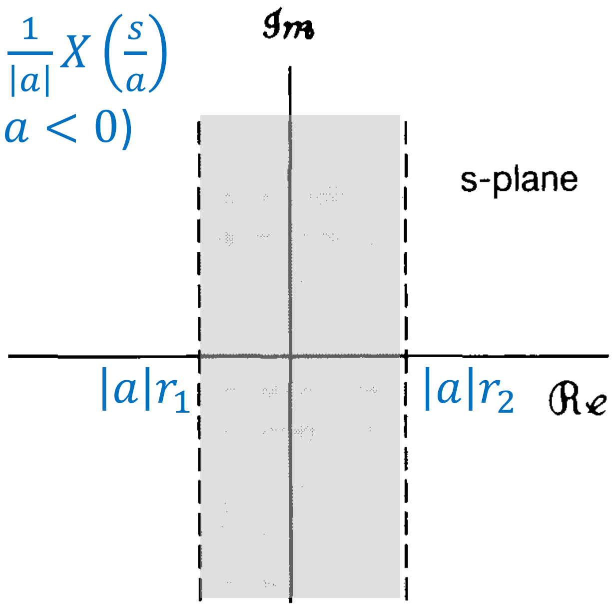
ROC of $X(s)$



ROC of $\frac{1}{|a|} X\left(\frac{s}{a}\right)$ $(0 < a < 1)$



ROC of $\frac{1}{|a|} X\left(\frac{s}{a}\right)$ $(-1 < a < 0)$



Properties of the Laplace transform

Conjugation

$$x(t) \xleftrightarrow{\mathfrak{L}} X(s) \quad \mathrm{ROC} = R$$

$$\Downarrow$$

$$x^*(t) \xleftrightarrow{\mathfrak{L}} X^*(s^*) \quad \mathrm{ROC} = R$$

$X(s) = X^*(s^*)$ if $x(t)$ is real

Convolution property

$$x_1(t) \xleftrightarrow{\mathfrak{L}} X_2(s) \quad \mathrm{ROC} = R_1$$

$$\Longrightarrow \quad x_1(t) * x_2(t) \xleftrightarrow{\mathfrak{L}} X_1(s)X_2(s)$$

$$x_2(t) \xleftrightarrow{\mathfrak{L}} X_2(s) \quad \mathrm{ROC} = R_2$$

ROC contains $R_1 \cap R_2$

Properties of the Laplace transform

Differentiation in the time domain

$$x(t) \stackrel{\mathfrak{L}}{\longleftrightarrow} X(s) \qquad \mathrm{ROC} = R$$

$$\Downarrow$$

$$\frac{dx(t)}{dt} \stackrel{\mathfrak{L}}{\longleftrightarrow} sX(s) \qquad \mathrm{ROC}\ \text{contains}\ R$$

$$x(t) = \frac{1}{2\pi j}\int_{\sigma-j\infty}^{\sigma+j\infty} X(s)e^{st}ds$$

$$\frac{dx(t)}{dt} = \frac{1}{2\pi j}\int_{\sigma-j\infty}^{\sigma+j\infty} sX(s)e^{st}ds$$

Differentiation in the s-domain

$$x(t) \stackrel{\mathfrak{L}}{\longleftrightarrow} X(s) \qquad \mathrm{ROC} = R$$

$$\Downarrow$$

$$-tx(t) \stackrel{\mathfrak{L}}{\longleftrightarrow} \frac{dX(s)}{ds} \qquad \mathrm{ROC} = R$$

$$X(s) = \int_{-\infty}^{+\infty} x(t)e^{-st}dt$$

$$\frac{dX(s)}{ds} = \int_{-\infty}^{+\infty} (-t)x(t)e^{-st}dt$$

Examples

$$\frac{t^{n-1}}{(n-1)!}e^{-at}u(t) \overset{\mathfrak{L}}{\longleftrightarrow} ?$$

Solution

Consider $x(t) = te^{-at}u(t)$

$$e^{-at}u(t) \overset{\mathfrak{L}}{\longleftrightarrow} \frac{1}{s+a} \qquad \mathcal{R}e\{s\} > -a$$

$$te^{-at}u(t) \overset{\mathfrak{L}}{\longleftrightarrow} -\frac{d}{ds}\left[\frac{1}{s+a}\right] = \frac{1}{(s+a)^2} \qquad \mathcal{R}e\{s\} > -a$$

$$\frac{t^2}{2}e^{-at}u(t) \overset{\mathfrak{L}}{\longleftrightarrow} \frac{1}{(s+a)^3} \qquad \mathcal{R}e\{s\} > -a$$

$$\frac{t^{n-1}}{(n-1)!}e^{-at}u(t) \overset{\mathfrak{L}}{\longleftrightarrow} \frac{1}{(s+a)^n} \qquad \mathcal{R}e\{s\} > -a$$

Properties of the Laplace transform

Examples

$$X(s)=\frac{2s^2+5s+5}{(s+1)^2(s+2)},\qquad \mathcal{Re}\{s\}>-1\qquad x(t)=?$$

Solution

$$X(s)=\frac{2}{(s+1)^2}-\frac{1}{s+1}+\frac{3}{s+2},\qquad \mathcal{Re}\{s\}>-1$$

$$x(t)=[2te^{-t}-e^{-t}+3e^{-2t}]u(t)$$

Properties of the Laplace transform

Integration in the time domain

$$x(t) \overset{\mathfrak{L}}{\longleftrightarrow} X(s) \qquad \mathrm{ROC} = R$$

$$\Downarrow$$

$$\int_{-\infty}^{t} x(\tau)d\tau \overset{\mathfrak{L}}{\longleftrightarrow} \frac{1}{s}X(s) \qquad \text{ROC contains } R \cap \{\mathcal{R}e\{s\} > 0\}$$

Proof

$$\int_{-\infty}^{t} x(\tau)d\tau = u(t) * x(t)$$

$$u(t) \overset{\mathfrak{L}}{\longleftrightarrow} \frac{1}{s} \qquad \mathcal{R}e\{s\} > 0$$

$$u(t) * x(t) \overset{\mathfrak{L}}{\longleftrightarrow} \frac{1}{s}X(s) \qquad \text{ROC contains } R \cap \{\mathcal{R}e\{s\} > 0\}$$

Properties of the Laplace transform

The initial- and final-theorems

❑ Initial-value theorem

If

$x(t) = 0$ for $t < 0$,

$x(t)$ contains no impulses or higher order singularities at the origin,

Then,

$$x(0^+) = \lim_{s\to\infty} sX(s)$$

❑ Final-value theorem

If

$x(t) = 0$ for $t < 0$,

$x(t)$ has a finite limit as $t \to \infty$,

Then,

$$\lim_{t\to\infty} x(t) = \lim_{s\to 0} sX(s)$$

Properties of the Laplace transform

Summary

TABLE 9.1 PROPERTIES OF THE LAPLACE TRANSFORM

Section	Property	Signal	Laplace Transform	ROC
		$x(t)$	$X(s)$	R
		$x_1(t)$	$X_1(s)$	R_1
		$x_2(t)$	$X_2(s)$	R_2
9.5.1	Linearity	$ax_1(t) + bx_2(t)$	$aX_1(s) + bX_2(s)$	At least $R_1 \cap R_2$
9.5.2	Time shifting	$x(t - t_0)$	$e^{-st_0}X(s)$	R
9.5.3	Shifting in the s-Domain	$e^{s_0 t}x(t)$	$X(s - s_0)$	Shifted version of R (i.e., s is in the ROC if $s - s_0$ is in R)
9.5.4	Time scaling	$x(at)$	$\frac{1}{\lvert a \rvert}X\left(\frac{s}{a}\right)$	Scaled ROC (i.e., s is in the ROC if s/a is in R)
9.5.5	Conjugation	$x^*(t)$	$X^*(s^*)$	R
9.5.6	Convolution	$x_1(t) * x_2(t)$	$X_1(s)X_2(s)$	At least $R_1 \cap R_2$
9.5.7	Differentiation in the Time Domain	$\frac{d}{dt}x(t)$	$sX(s)$	At least R
9.5.8	Differentiation in the s-Domain	$-tx(t)$	$\frac{d}{ds}X(s)$	R
9.5.9	Integration in the Time Domain	$\int_{-\infty}^{t} x(\tau)d(\tau)$	$\frac{1}{s}X(s)$	At least $R \cap \{\mathcal{R}e\{s\} > 0\}$

Initial- and Final-Value Theorems

9.5.10 If $x(t) = 0$ for $t < 0$ and $x(t)$ contains no impulses or higher-order singularities at $t = 0$, then

$$x(0^+) = \lim_{s \to \infty} sX(s)$$

If $x(t) = 0$ for $t < 0$ and $x(t)$ has a finite limit as $t \longrightarrow \infty$, then

$$\lim_{t \to \infty} x(t) = \lim_{s \to \infty} sX(s)$$

The Laplace Transform
(ch.9)

- ❑ The Laplace transform
- ❑ The region of convergence for Laplace transforms
- ❑ The inverse Laplace transform
- ❑ Geometric evaluation of the Fourier transform from the pole-zero plot
- ❑ Properties of the Laplace transform
- ❑ Some Laplace transform pairs
- ❑ Analysis and characterization of LTI systems using the Laplace transform
- ❑ System function algebra and block diagram representations
- ❑ The unilateral Laplace transform

Some Laplace transform pairs

TABLE 9.2 LAPLACE TRANSFORMS OF ELEMENTARY FUNCTIONS

Transform pair	Signal	Transform	ROC
1	$\delta(t)$	1	All s
2	$u(t)$	$\frac{1}{s}$	$\Re e\{s\} > 0$
3	$-u(-t)$	$\frac{1}{s}$	$\Re e\{s\} < 0$
4	$\frac{t^{n-1}}{(n-1)!}u(t)$	$\frac{1}{s^n}$	$\Re e\{s\} > 0$
5	$-\frac{t^{n-1}}{(n-1)!}u(-t)$	$\frac{1}{s^n}$	$\Re e\{s\} < 0$
6	$e^{-\alpha t}u(t)$	$\frac{1}{s+\alpha}$	$\Re e\{s\} > -\alpha$
7	$-e^{-\alpha t}u(-t)$	$\frac{1}{s+\alpha}$	$\Re e\{s\} < -\alpha$
8	$\frac{t^{n-1}}{(n-1)!}e^{-\alpha t}u(t)$	$\frac{1}{(s+\alpha)^n}$	$\Re e\{s\} > -\alpha$
9	$-\frac{t^{n-1}}{(n-1)!}e^{-\alpha t}u(-t)$	$\frac{1}{(s+\alpha)^n}$	$\Re e\{s\} < -\alpha$
10	$\delta(t-T)$	e^{-sT}	All s
11	$[\cos \omega_0 t]u(t)$	$\frac{s}{s^2+\omega_0^2}$	$\Re e\{s\} > 0$
12	$[\sin \omega_0 t]u(t)$	$\frac{\omega_0}{s^2+\omega_0^2}$	$\Re e\{s\} > 0$
13	$[e^{-\alpha t}\cos \omega_0 t]u(t)$	$\frac{s+\alpha}{(s+\alpha)^2+\omega_0^2}$	$\Re e\{s\} > -\alpha$
14	$[e^{-\alpha t}\sin \omega_0 t]u(t)$	$\frac{\omega_0}{(s+\alpha)^2+\omega_0^2}$	$\Re e\{s\} > -\alpha$
15	$u_n(t) = \frac{d^n\delta(t)}{dt^n}$	s^n	All s
16	$u_{-n}(t) = \underbrace{u(t) * \cdots * u(t)}_{n \text{ times}}$	$\frac{1}{s^n}$	$\Re e\{s\} > 0$

The Laplace Transform
(ch.9)

- ❑ The Laplace transform
- ❑ The region of convergence for Laplace transforms
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- ❑ System function algebra and block diagram representations
- ❑ The unilateral Laplace transform

Analysis and characterization of LTI systems using the Laplace transform

$$e^{st} \longrightarrow \boxed{\text{LTI}} \longrightarrow y(t) = H(s)e^{st}$$

$$H(s) = \int_{-\infty}^{\infty} h(\tau)e^{-s\tau}d\tau$$

$$x(t) \longrightarrow \boxed{\text{LTI}} \longrightarrow y(t) = x(t) * h(t)$$

$$Y(s) = X(s)\,H(s)$$

$H(s)$: system function or transfer function

Analysis and characterization of LTI systems using the Laplace transform

Causality

Causal $\Longrightarrow$ ROC of $H(s)$ is a right-half plane Converse is not necssaryily true

A system with rational $H(s)$ is causal $\Longleftrightarrow$ ROC of $H(s)$ is the right-half plane to the right of the right-most pole

Examples $h(t) = e^{-t}u(t)$ Causal?

Solution 1

$h(t) = 0$ for $t < 0$

$\Longrightarrow$ Causal

Solution 2

$$H(s) = \frac{1}{s+1} \quad \mathcal{Re}\{s\} > -1$$

$\Longrightarrow$ Causal

Examples $h(t) = e^{-|t|}$ Causal?

Solution 1

$h(t) \neq 0$ for $t < 0$

$\Longrightarrow$ Noncausal

Solution 2

$$H(s) = \frac{-2}{s^2-1} \quad -1 < \mathcal{Re}\{s\} < 1$$

$\Longrightarrow$ Noncausal

Examples

$$H(s) = \frac{e^s}{s+1}, \quad \mathcal{Re}\{s\} > -1 \qquad \text{Causal?}$$

Solution

$$e^{-t}u(t) \xleftrightarrow{\mathfrak{L}} \frac{1}{s+1} \quad \mathcal{Re}\{s\} > -1$$

Time-shifting

$$e^{-(t+1)}u(t+1) \xleftrightarrow{\mathfrak{L}} \frac{e^s}{s+1} \quad \mathcal{Re}\{s\} > -1$$

$$x(t) \xleftrightarrow{\mathfrak{L}} X(s) \qquad \mathrm{ROC} = R$$

$$\Downarrow$$

$$x(t-t_0) \xleftrightarrow{\mathfrak{L}} e^{-st_0}X(s) \qquad \mathrm{ROC} = R$$

$$h(t) = e^{-(t+1)}u(t+1)$$

$\Longrightarrow$ Noncausal

Anti-causality

Anti-causal $\Longrightarrow$ ROC of $H(s)$ is a left-half plane   Converse is not necssaryily true

A system with rational $H(s)$ is anti-causal $\Longleftrightarrow$ ROC of $H(s)$ is the left-half plane to the left of the left-most pole

Stability

Stable $\Leftrightarrow$ The impulse response of $H(s)$ is absolutely integrable

$\Downarrow$

Stable $\Leftrightarrow$ The ROC of $H(s)$ includes the entire $j\omega$-axis

Examples

$$H(s) = \frac{s-1}{(s+1)(s-2)}$$

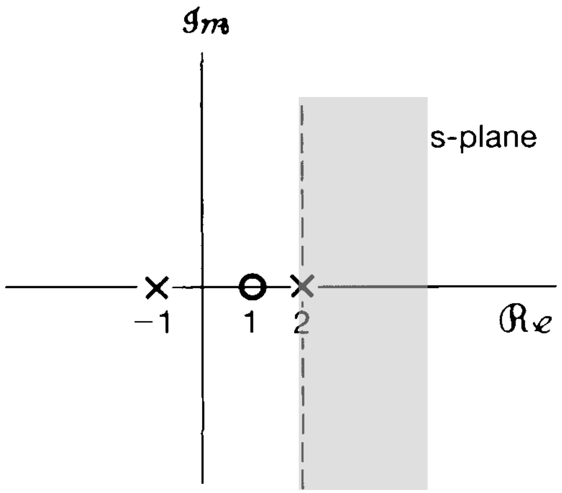
Causal? Stable?

Solution

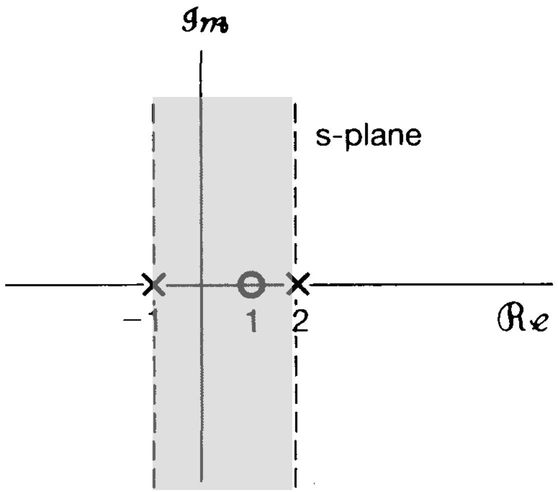
$$h(t) = \left(\frac{2}{3}e^{t} + \frac{1}{3}e^{2t}\right)u(t)$$

$$h(t) = \frac{2}{3}e^{-t}u(t) - \frac{1}{3}e^{2t}u(-t)$$

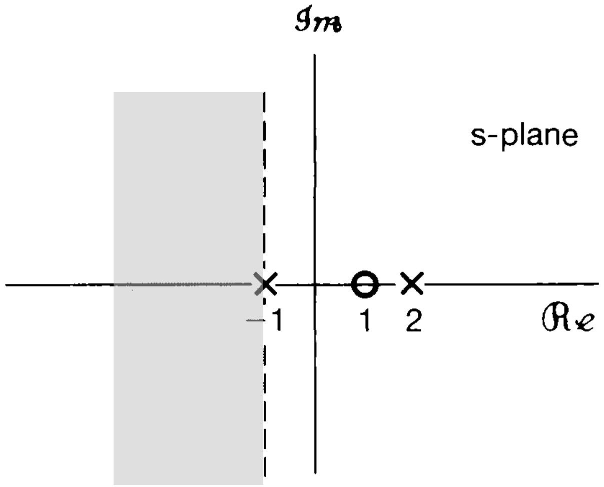
$$h(t) = -\left(\frac{2}{3}e^{t} + \frac{1}{3}e^{2t}\right)u(-t)$$



Causal
Unstable system



Noncausal
Stable system



Anti-causal
Unstable system

Analysis and characterization of LTI systems using the Laplace transform

Stability

For a causal system, with rational system function $H(s)$,

Stable $\Longleftrightarrow$ All the poles of $H(s)$ lie in the left-half of the s-plane

OR

Stable $\Longleftrightarrow$ All the poles have negative real parts

Examples

$$H(s) = \frac{1}{(s+1)}$$

Pole: $s = -1$

Causal $\Longrightarrow$ Stable

$$H(s) = \frac{1}{(s-2)}$$

Pole: $s = 2$

Causal $\Longrightarrow$ Unstable

Examples

Consider the class of causal second-order systems

$$h(t) = M[e^{c_1 t} - e^{c_2 t}]u(t)$$

$$H(s) = \frac{\omega_n}{s^2 + 2\zeta\omega_n s + \omega_n^2} = \frac{\omega_n}{(s - c_1)(s - c_2)}$$

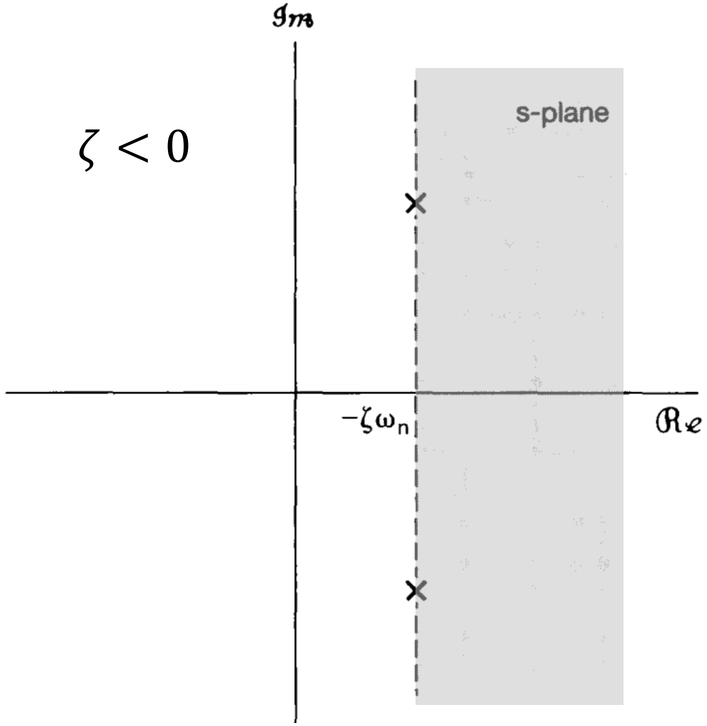
$$c_1 = -\zeta\omega_n + \omega_n\sqrt{\zeta^2 - 1}$$

$$c_2 = -\zeta\omega_n - \omega_n\sqrt{\zeta^2 - 1}$$

$$M = \frac{\omega_n}{2\sqrt{\zeta^2 - 1}}$$

Is the system stable when $\zeta < 0$?

$\zeta < 0$



Solution

Unstable

LTI systems characterized by linear constant-coefficient differential equations

❑ Examples

$$\frac{dy(t)}{dt} + 3y(t) = x(t)$$

$$sY(s) + 3Y(s) = X(s)$$

$$H(s) = \frac{1}{s+3}$$

Differential equation: not a complete specification of the LTI system!

Pre-knowledge: if causal $\quad h(t) = e^{-3t}u(t)$

Anti-causal $\quad h(t) = -e^{-3t}u(-t)$

LTI systems characterized by linear constant-coefficient differential equations

❑ Generally

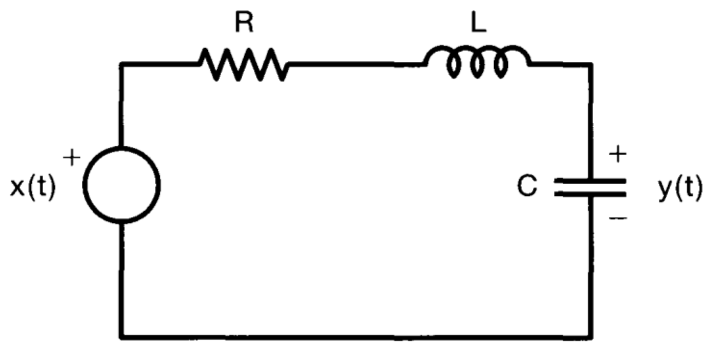
$$\sum_{k=0}^{N} a_k \frac{d^k y(t)}{dt^k} = \sum_{k=0}^{M} b_k \frac{d^k x(t)}{dt^k}$$

$$\left(\sum_{k=0}^{N} a_k s^k\right) Y(s) = \left(\sum_{k=0}^{M} b_k s^k\right) X(s)$$

$$H(s) = \frac{\sum_{k=0}^{M} b_k s^k}{\sum_{k=0}^{N} a_k s^k} \Longrightarrow \begin{cases} \text{Poles at the solution of } \displaystyle\sum_{k=0}^{N} a_k s^k = 0 \\ \text{Zeros at the solution of } \displaystyle\sum_{k=0}^{M} b_k s^k = 0 \end{cases}$$

Examples

$$RC\frac{dy(t)}{dt} + LC\frac{d^2y(t)}{dt^2} + y(t) = x(t)$$



Solution

$$H(s) = \frac{1/LC}{s^2 + (R/L)s + (1/LC)}$$

Poles have negative real parts when $R > 0$, $L > 0$, and $C > 0$

$\Longrightarrow$ Stable

Examples relating system behavior to the system function

If the input to an LTI system is $x(t) = e^{-3t}u(t)$

Then the output is $y(t) = [e^{-t} - e^{-2t}]u(t)$

System function?

Solution

$$X(s) = \frac{1}{s+3}, \qquad \mathcal{Re}\{s\} > -3$$

$$Y(s) = \frac{1}{(s+1)(s+2)}, \qquad \mathcal{Re}\{s\} > -1$$

$$H(s) = \frac{Y(s)}{X(s)} = \frac{s+3}{(s+1)(s+2)} = \frac{s+3}{s^2+3s+2}$$

Causal and stable

$$\frac{d^2y(t)}{dt^2} + 3\frac{dy(t)}{dt} + 2y(t) = \frac{dx(t)}{dt} + 3x(t)$$

Examples relating system behavior to the system function

Given the following information about an LTI system, determine $H(s)$.

1. The system is causal;
2. $H(s)$ is rational and has only two poles at $s = -2$ and $s = 4$;
3. If $x(t) = 1$, then $y(t) = 0$;
4. $h(0^+) = 4$

Solution

$$H(s) = \frac{p(s)}{(s+2)(s-4)} = \frac{p(s)}{s^2 - 2s - 8}$$ $p(s)$ is an polynomial in s

$$p(0) = 0 \quad \Longrightarrow \quad p(s) = sq(s)$$ $q(s)$ is an polynomial in s

$$\lim_{s\to\infty} sH(s) = \lim_{s\to\infty} \frac{s^2 q(s)}{s^2 - 2s - 8} = \lim_{s\to\infty} \frac{Ks^2}{s^2 - 2s - 8} = 4$$ $q(s) = K$ is a constant

$$K = 4 \quad \Longrightarrow \quad H(s) = \frac{4s}{(s+2)(s-4)}, \qquad \mathcal{Re}\{s\} > 4$$

Examples relating system behavior to the system function

A stable and causal system with impulse response $h(t)$ and system function $H(s)$, which is rational and contains a pole at s=-2, and does not have a zero at the origin.

- ❑ $\mathcal{F}\{h(t)e^{3t}\}$ converges. False
- ❑ $\int_{-\infty}^{+\infty} h(t)dt = 0$ False
- ❑ $th(t)$ is the impulse response of a causal and stable system. True
- ❑ $dh(t)/dt$ contains at least one pole in its Laplace transform. True
- ❑ $h(t)$ has finite duration. False
- ❑ $H(s) = H(-s)$. False
- ❑ $\lim_{s\to\infty} H(s) = 2.$ Insufficient information

The Laplace Transform
(ch.9)

- ❑ The Laplace transform
- ❑ The region of convergence for Laplace transforms
- ❑ The inverse Laplace transform
- ❑ Geometric evaluation of the Fourier transform from the pole-zero plot
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- ❑ Some Laplace transform pairs
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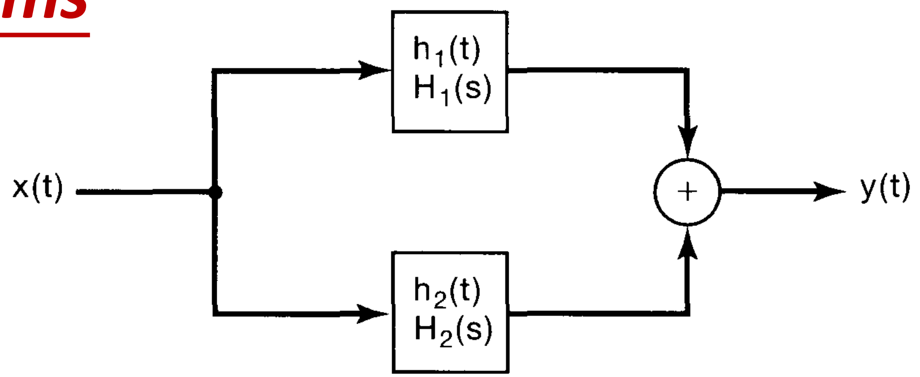
- ❑ System function algebra and block diagram representations
- ❑ The unilateral Laplace transform

System functions for interconnections of LTI systems

- Parallel interconnection

$$h(t) = h_1(t) + h_2(t)$$

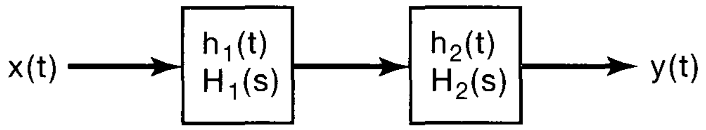
$$H(s) = H_1(s) + H_2(s)$$



- Series interconnection

$$h(t) = h_1(t) * h_2(t)$$

$$H(s) = H_1(s)H_2(s)$$



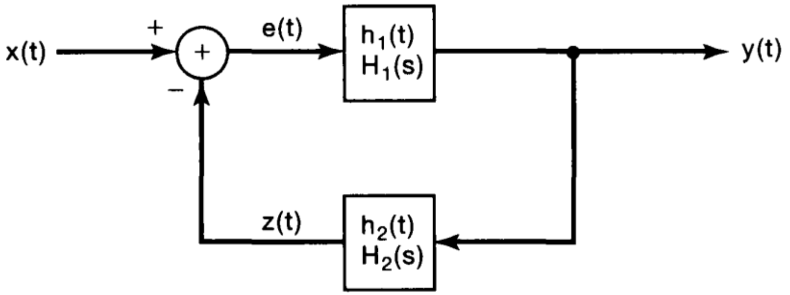
- Feedback interconnection

$$Y(s) = H_1(s)E(s)$$

$$E(s) = X(s) - Z(s)$$

$$Z(s) = H_2(s)Y(s)$$

$$H(s) = \frac{H_1(s)}{1 + H_1(s)H_2(s)}$$

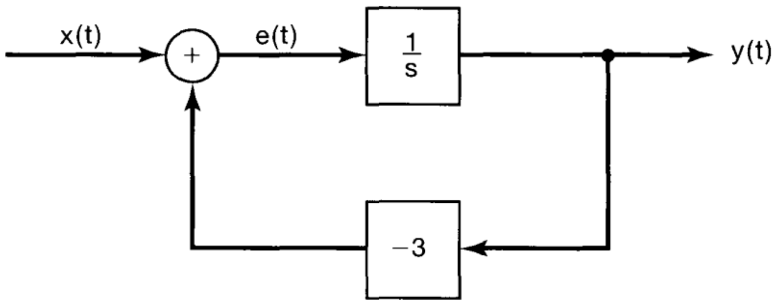


Block diagram representations for causal LTI systems

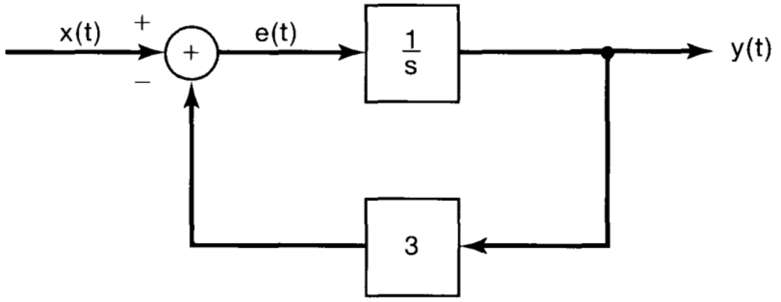
$$H(s) = \frac{1}{s+3}$$

$$\frac{dy(t)}{dt} + 3y(t) = x(t)$$

$$H(s) = \frac{1/s}{1 + 3/s}$$



Or equivalently



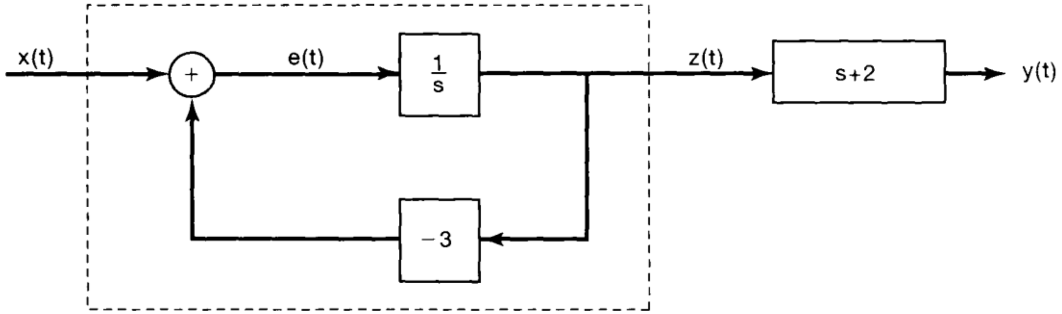
Using basic operations: addition, multiplication, and integration

Examples: block diagram representations for causal LTI systems

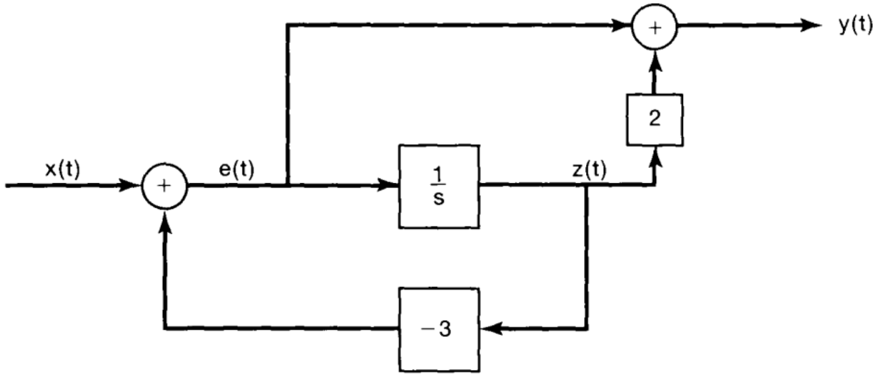
$$H(s) = \frac{s+2}{s+3} = \left(\frac{1}{s+3}\right)(s+2)$$

$$y(t) = \frac{dz(t)}{dt} + 2z(t)$$

$$y(t) = e(t) + 2z(t)$$



Or equivalently

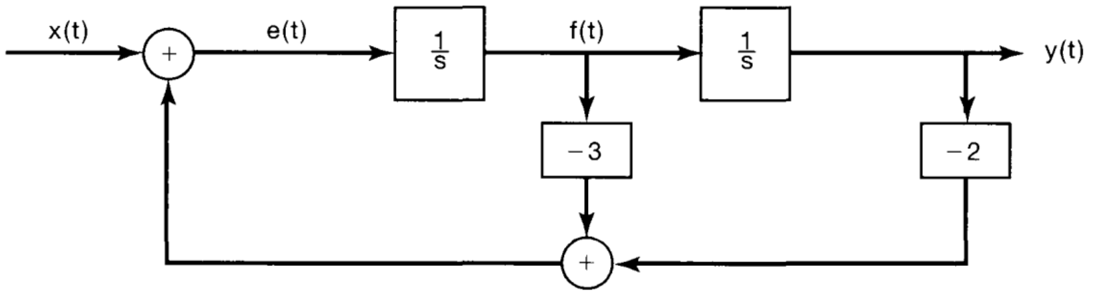


Examples: block diagram representations for causal LTI systems

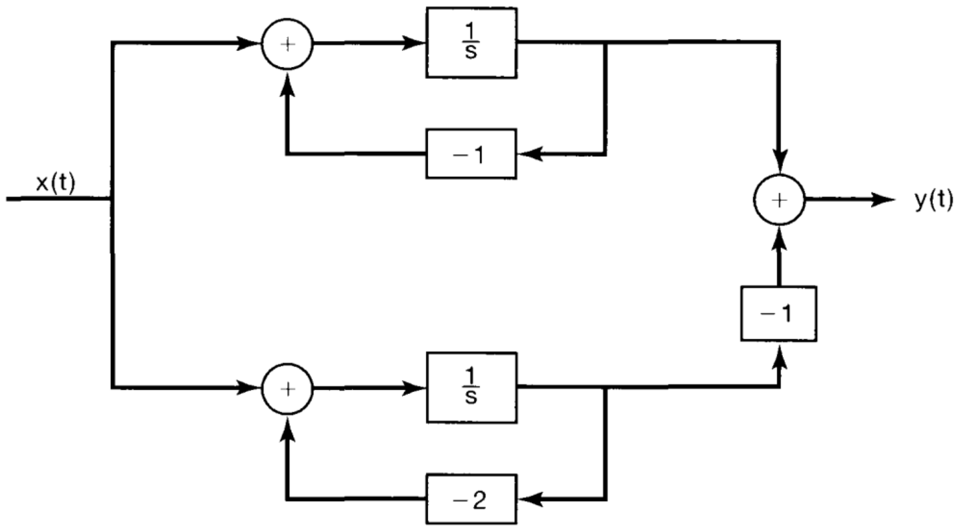
$$H(s) = \frac{1}{s^2 + 3s + 2} = \frac{1}{(s+1)} \cdot \frac{1}{(s+2)} = \frac{1}{(s+1)} - \frac{1}{(s+2)}$$

$$\frac{d^2 y(t)}{dt^2} + 3\frac{dy(t)}{dt} + 2y(t) = x(t)$$

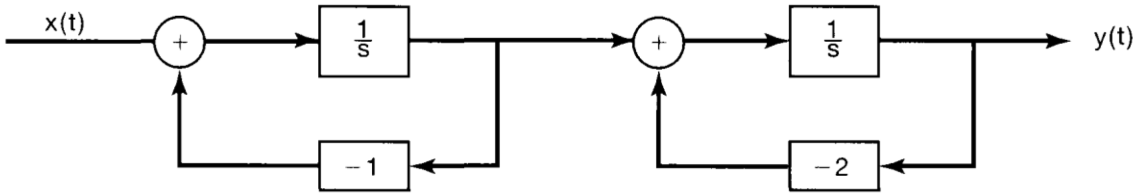
Direct form $e(t) = \frac{d^2 y(t)}{dt^2}$



Parallel form



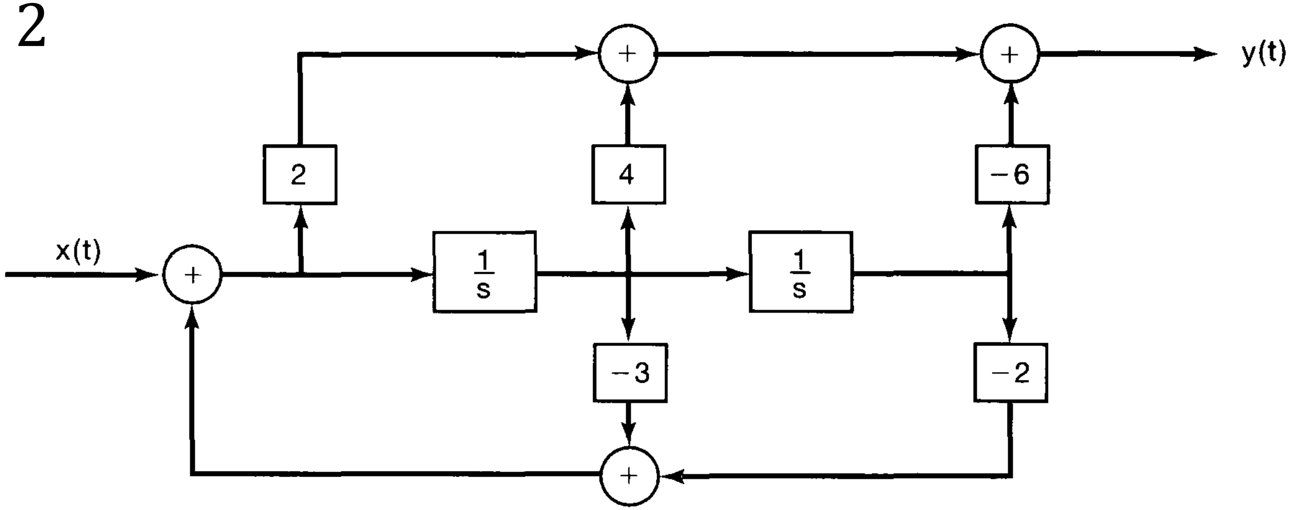
Cascade form



Examples: block diagram representations for causal LTI systems

$$H(s) = \frac{2s^2 + 4s - 6}{s^2 + 3s + 2}$$

Direct form



Parallel form

$$H(s) = 2 + \frac{6}{s+2} - \frac{8}{s+1}$$

Cascade form

$$H(s) = \left(\frac{2(s-1)}{s+2}\right)\left(\frac{s+3}{s+1}\right)$$

$$H(s) = \left(\frac{s+3}{s+2}\right)\left(\frac{2(s-1)}{s+1}\right)$$

The Laplace Transform
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$$x(t) \xleftrightarrow{\mathcal{UL}} \mathcal{X}(s) = \mathcal{UL}\{x(t)\}$$

$$\mathcal{X}(s) \triangleq \int_{0^-}^{\infty} x(t)e^{-st}dt$$

Examples

$$x(t) = \frac{t^{n-1}}{(n-1)!}e^{-at}u(t)$$

$x(t) = 0$, for $t < 0$

$$\mathcal{X}(s) \triangleq \frac{1}{(s+a)^n}, \qquad \mathcal{Re}\{s\} > -a$$

- $x(t) = 0$, for $t < 0$, the unilateral and bilateral transforms are identical

Examples

$$x(t) = e^{-a(t+1)}u(t+1)$$

$$X(s) = \frac{e^{s}}{s+a}, \qquad \mathcal{R}e\{s\} > -a$$

$$\mathcal{X}(s) = \int_{0^-}^{\infty} e^{-a(t+1)}u(t+1)e^{-st}dt$$

$$= \int_{0^-}^{\infty} e^{-a}e^{-t(s+a)}dt$$

$$= \frac{e^{-a}}{s+a}, \qquad \mathcal{R}e\{s\} > -a$$

- $x(t) \neq 0$, for $-1 < t < 0$, the unilateral and bilateral transforms are different

Examples

$$x(t) = \delta(t) + 2u_1(t) + e^t u(t)$$

$x(t) = 0$, for $t < 0$

$$\mathcal{X}(s) = X(s)$$

$$= 1 + 2s + \frac{1}{s-1}$$

$$= \frac{s(2s-1)}{s-1}, \qquad \mathcal{Re}\{s\} > 1$$

Examples

$$\mathcal{X}(s) = \frac{1}{(s+1)(s+2)}, \qquad \mathcal{Re}\{s\} > -1$$

$$x(t) = [e^{-t} - e^{-2t}]u(t) \quad \text{for } t > 0^-$$

Examples

$$\mathcal{X}(s) = \frac{s^2 - 3}{s + 2}$$

$$= -2 + s + \frac{1}{s+2}, \qquad \mathcal{Re}\{s\} > -2$$

$$x(t) = -2\delta(t) + u_1(t) + e^{-2t}u(t) \quad \text{for } t > 0^-$$

Note: $u_n(t) = \dfrac{d^n\delta(t)}{dt^n}$

The unilateral Laplace transform

Properties of the unilateral Laplace transform

Property	Signal	Unilateral Laplace Transform
	$x(t)$ $x_1(t)$ $x_2(t)$	$\mathfrak{X}(s)$ $\mathfrak{X}_1(s)$ $\mathfrak{X}_2(s)$
Linearity	$ax_1(t) + bx_2(t)$	$a\mathfrak{X}_1(s) + b\mathfrak{X}_2(s)$
Shifting in the s-domain	$e^{s_0 t}x(t)$	$\mathfrak{X}(s - s_0)$
Time scaling	$x(at), \quad a > 0$	$\frac{1}{a}\mathfrak{X}\left(\frac{s}{a}\right)$
Conjugation	$x*(t)$	$x*(s)$
$\boxed{\text{Convolution (assuming that } x_1(t) \text{ and } x_2(t) \text{ are identically zero for } t < 0)}$	$x_1(t) * x_2(t)$	$\mathfrak{X}_1(s)\mathfrak{X}_2(s)$

Property	Signal	Unilateral Laplace Transform
Differentiation in the time domain	$\frac{d}{dt}x(t)$	$\boxed{s\mathfrak{X}(s) - x(0^-)}$
Differentiation in the s-domain	$-tx(t)$	$\frac{d}{ds}\mathfrak{X}(s)$
Integration in the time domain	$\int_{0^-}^{t} x(\tau)\,d\tau$	$\frac{1}{s}\mathfrak{X}(s)$

Initial- and Final-Value Theorems

If $x(t)$ contains no impulses or higher-order singularities at $t = 0$, then

$$x(0^+) = \lim_{s\to\infty} s\mathfrak{X}(s)$$

$$\lim_{t\to\infty} x(t) = \lim_{s\to 0} s\mathfrak{X}(s)$$

Note: no ROC is specified cause it is always the right-half plane

Differentiation property

$$x(t) \xleftrightarrow{\mathcal{UL}} \mathcal{X}(s) \qquad \frac{dx(t)}{dt} \xleftrightarrow{\mathcal{UL}} s\mathcal{X}(s) - x(0^-)$$

$$\int_{0^-}^{\infty} \frac{dx(t)}{dt} e^{-st} dt = x(t)e^{-st}\Big|_{0^-}^{\infty} + s\int_{0^-}^{\infty} x(t)e^{-st} dt = s\mathcal{X}(s) - x(0^-)$$

Similarly

$$\frac{d^2x(t)}{dt^2} \xleftrightarrow{\mathcal{UL}} s^2\mathcal{X}(s) - sx(0^-) - x'(0^-)$$

Convolution property

$$x_1(t) \overset{\mathcal{UL}}{\longleftrightarrow} \mathcal{X}_1(s)$$
$$x_2(t) \overset{\mathcal{UL}}{\longleftrightarrow} \mathcal{X}_2(s)$$

$$x_1(t) * x_2(t) \overset{\mathcal{UL}}{\longleftrightarrow} \mathcal{X}_1(s)\mathcal{X}_2(s)$$

Only if $x_1(t)$ and $x_2(t)$ are zero for $t < 0$

- Example

A causal LTI system: $\frac{d^2y(t)}{dt^2} + 3\frac{dy(t)}{dt} + 2y(t) = x(t)$

If: $x(t) = \alpha u(t), y(t) =?$

- Solution

Causal $\Rightarrow$ $\mathcal{H}(s) = H(s) = \frac{1}{s^2+3s+2}$

$$\mathcal{Y}(s) = \mathcal{H}(s)\mathcal{X}(s) = \frac{\alpha}{s(s+1)(s+2)} = \frac{\alpha/2}{s} - \frac{\alpha}{s+1} + \frac{\alpha/2}{s+2}$$

convolution property for unilateral Laplace transforms

$$\therefore\ y(t) = \alpha\left[\frac{1}{2} - e^{-t} + \frac{1}{2}e^{-2t}\right]u(t)$$

Note: this can also be done by bilateral Laplace transforms

Solving differential equations using the unilateral Laplace transform

- Why unilateral Laplace transform? Non-zero initial condition
- Example: A LTI system: $\frac{d^2y(t)}{dt^2} + 3\frac{dy(t)}{dt} + 2y(t) = x(t)$ $\quad y(0^-) = \beta \quad y'(0^-) = \gamma$

  If $x(t) = \alpha u(t), y(t) = ?$
- Solution

$$s^2\mathcal{Y}(s) - \beta s - \gamma + 3s\mathcal{Y}(s) - 3\beta + 2\mathcal{Y}(s) = \frac{\alpha}{s}$$

$$\mathcal{Y}(s) = \frac{\beta(s+3)}{(s+1)(s+2)} + \frac{\gamma}{(s+1)(s+2)} + \frac{\alpha}{s(s+1)(s+2)}$$

$$y(t) = \mathcal{U}\mathfrak{L}^{-1}\{\mathcal{Y}(s)\}$$

Solving differential equations using the unilateral Laplace transform

- **Example:** A LTI system: $\frac{d^2y(t)}{dt^2} + 3\frac{dy(t)}{dt} + 2y(t) = x(t) \qquad y(0^-) = \beta \quad y'(0^-) = \gamma$

  If $x(t) = \alpha u(t), y(t) =?$

$$\mathcal{Y}(s) = \boxed{\frac{\beta(s+3)}{(s+1)(s+2)} + \frac{\gamma}{(s+1)(s+2)}} + \boxed{\frac{\alpha}{s(s+1)(s+2)}}$$

$$\Big\updownarrow \mathcal{UL}^{-1} \qquad\qquad \Big\updownarrow \mathcal{UL}^{-1}$$

$$y(t) = \qquad y_1(t) \qquad + \qquad y_2(t)$$

Zero-input response　　　　Zero-state response