# Sampling (ch.7)

- ❑ Representation of a Continuous-Time Signal by Its Samples: The Sampling Theorem
- ❑ Reconstruction of a Signal from Its Samples Using Interpolation
- ❑ The Effect of Undersampling: Aliasing
- ❑ Discrete-Time Processing of Continuous-Time Signals
- ❑ Sampling of Discrete-Time Signals

# The Sampling Theorem

- What is sampling?

  Converting continuous-time signals to discrete-time signals
- Why sampling?

  To use the well-developed digital technology
- But, a signal could not always be uniquely specified by equally-spaced samples

$$x_1(kT) = x_2(kT) = x_3(kT)$$

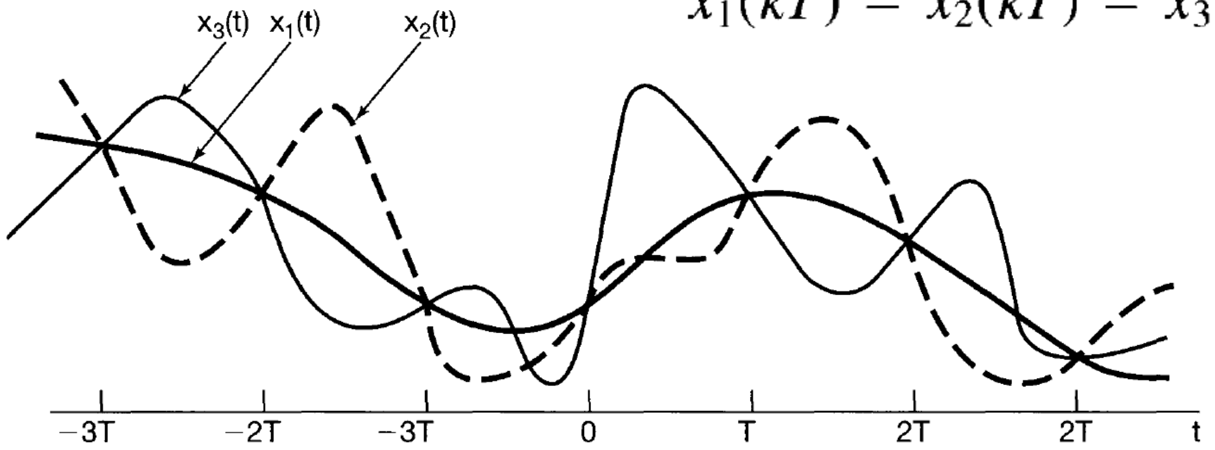
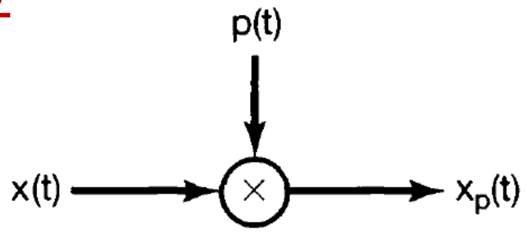



**Figure 7.1** Three continuous-time signals with identical values at integer multiples of *T*.

- The sampling theorem should be satisfied

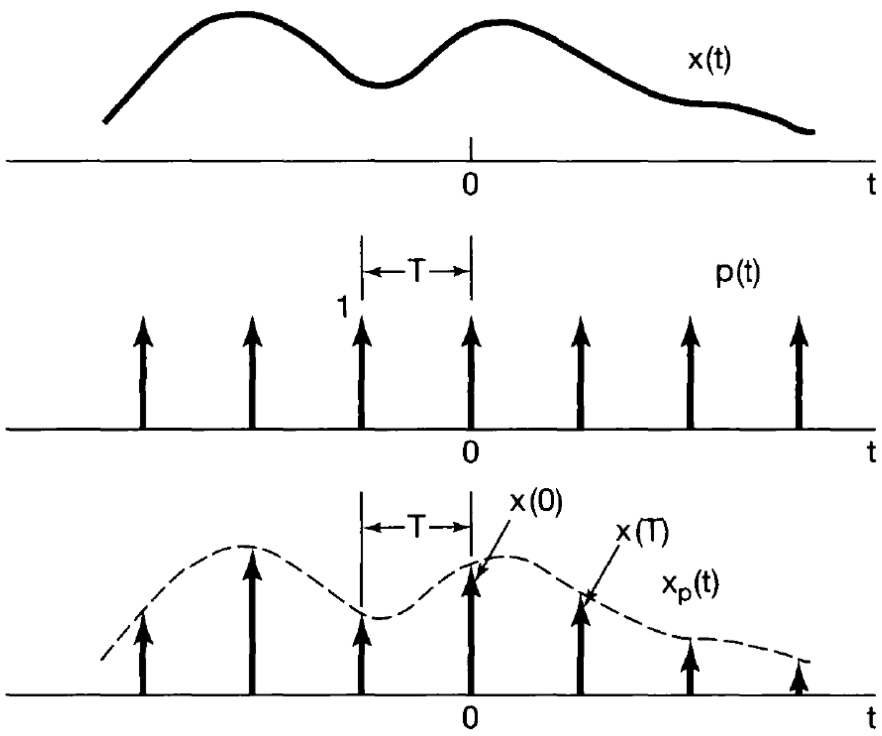
# The Sampling Theorem

## *Impulse-Train Sampling*




$$x_p(t) = x(t) \cdot p(t)$$

- Time domain

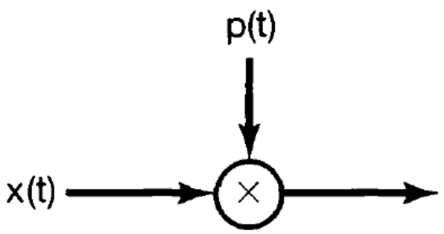



$$p(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT)$$

$$x_p(t) = \sum_{n=-\infty}^{\infty} x(nT) \cdot \delta(t - nT)$$

# The Sampling Theorem

## Impulse-Train Sampling



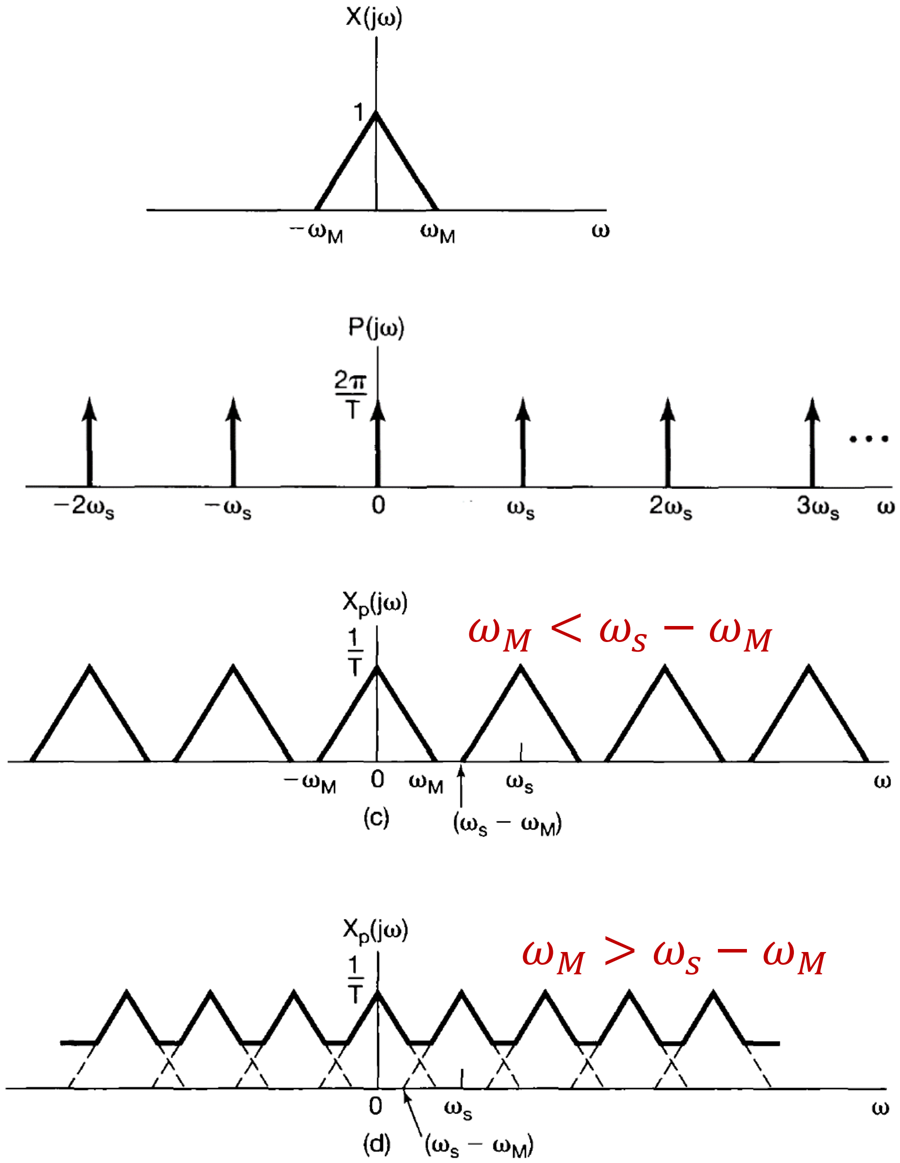

$$x_p(t) = x(t) \cdot p(t)$$

- Frequency domain

$$X_p(j\omega) = \frac{1}{2\pi} X(j\omega) * P(j\omega)$$

$$P(j\omega) = \frac{2\pi}{T} \sum_{K=-\infty}^{\infty} \delta(\omega - k\omega_s) = \frac{2\pi}{T} \sum_{K=-\infty}^{\infty} \delta\left(\omega - k\frac{2\pi}{T}\right)$$

$$X_p(j\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\theta) P(j(\omega - \theta)) d\theta = \frac{1}{T} \sum_{K=-\infty}^{\infty} X\big(j(\omega - k \cdot \omega_s)\big)$$



# The Sampling Theorem

## Sampling Theorem

Sampling Theorem:
Let $x(t)$ be a band-limited signal with $X(j\omega) = 0$ for $|\omega| > \omega_M$. Then $x(t)$ is uniquely determined by its samples $x(nT)$, $n = 0, \pm 1, \pm 2, \ldots$, if

$$\omega_s > 2\omega_M,$$

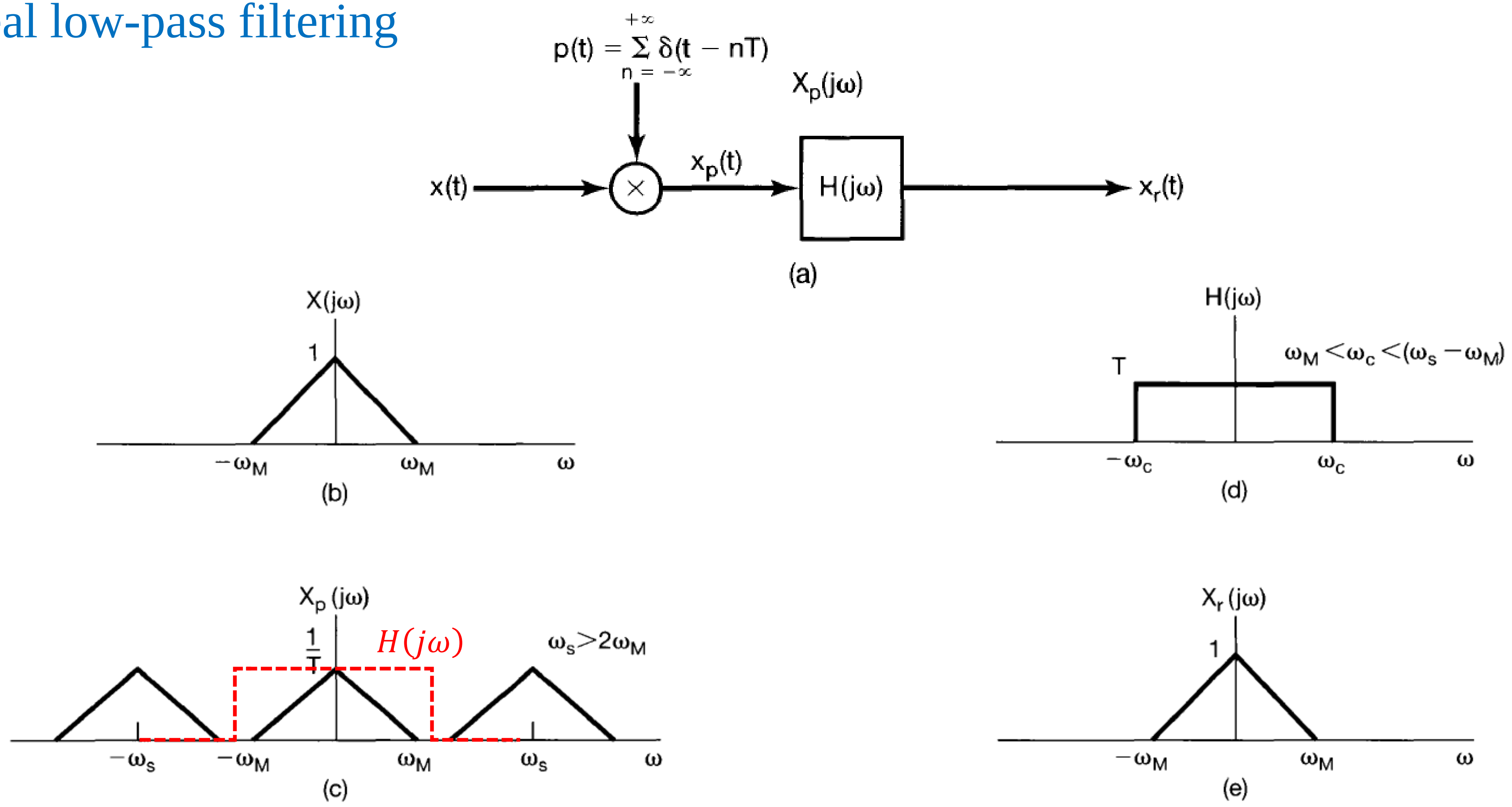
where

$$\omega_s = \frac{2\pi}{T}.$$

Given these samples, we can reconstruct $x(t)$ by generating a periodic impulse train in which successive impulses have amplitudes that are successive sample values. This impulse train is then processed through an ideal lowpass filter with gain $T$ and cutoff frequency greater than $\omega_M$ and less than $\omega_s - \omega_M$. The resulting output signal will exactly equal $x(t)$.

## *Recovery of the CT signal*

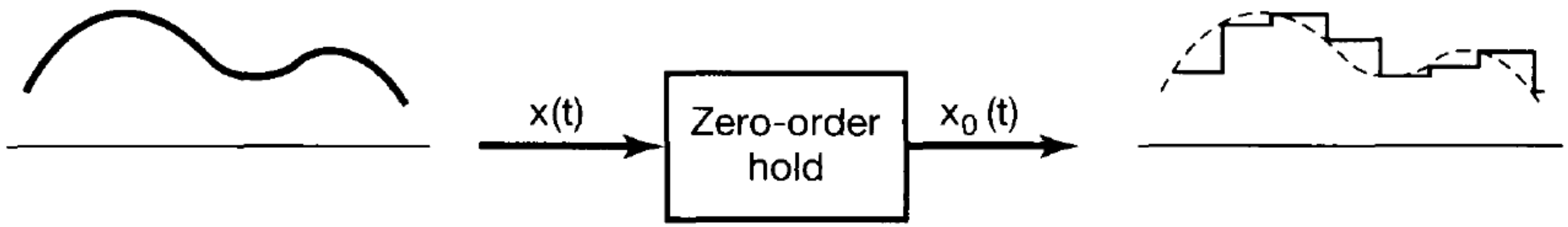
- Ideal low-pass filtering



# The Sampling Theorem

## *Sampling with a Zero-order Hold*

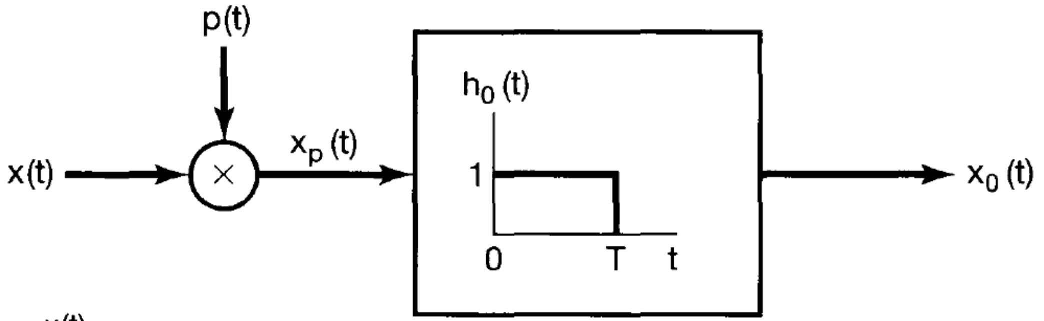
- Why: Impulse-train is difficult to generate
- Principle: Samples $x(t)$ at a given instant and holds that value until the next instant

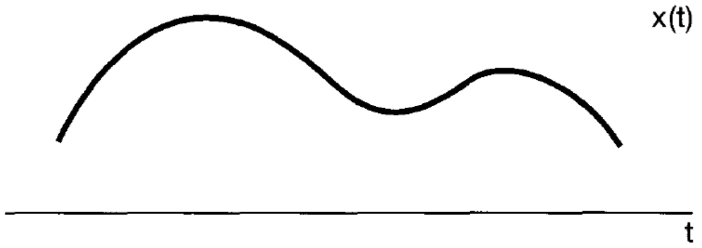


# The Sampling Theorem

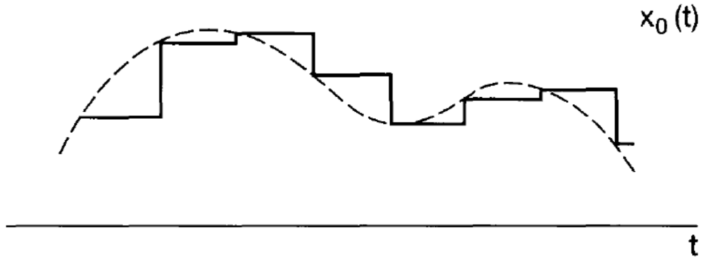
## *Sampling with a Zero-order Hold*

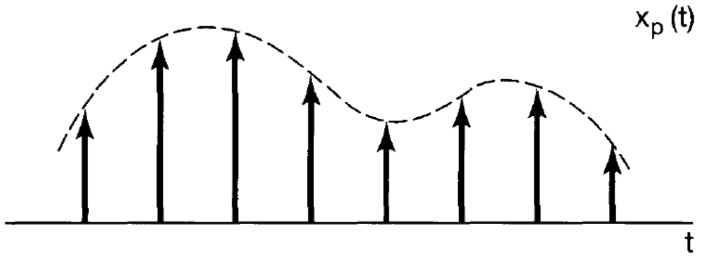
- Equivalent: Impulse-train sampling + an LTI system with a rectangular impulse response




$$x_0(t) = x_p(t) * h(t) \Longrightarrow$$

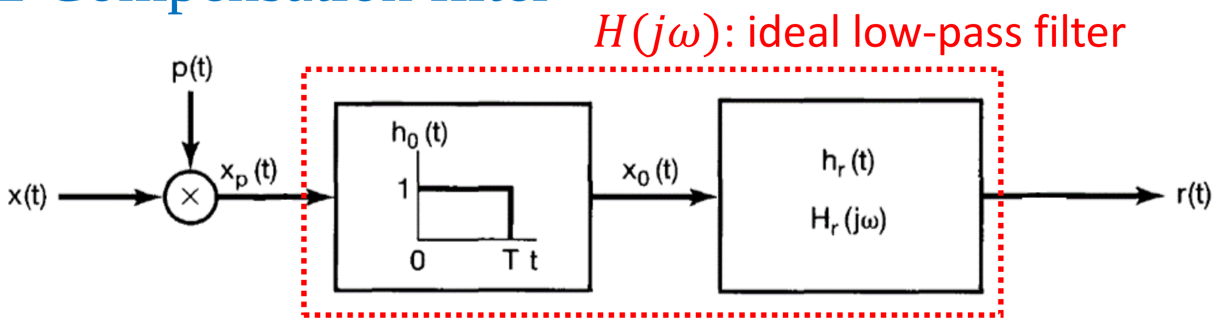


# The Sampling Theorem

## ***Sampling with a Zero-order Hold***

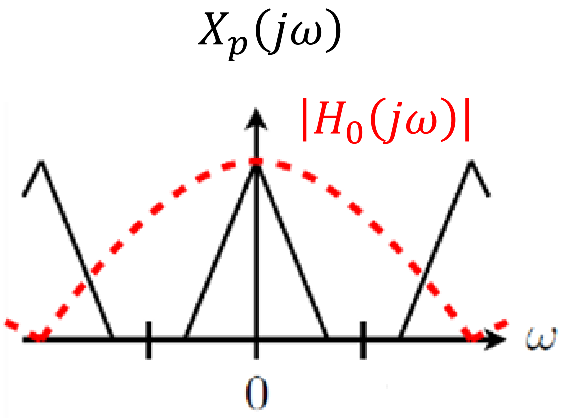
❑ Compensation filter

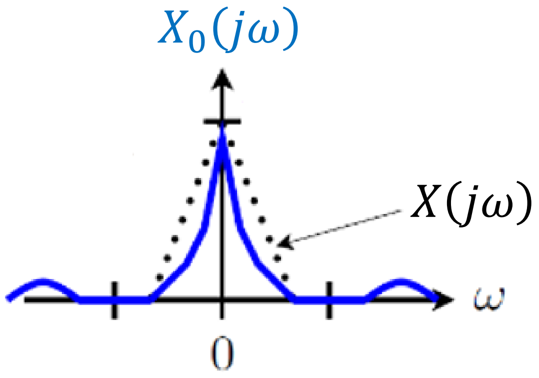


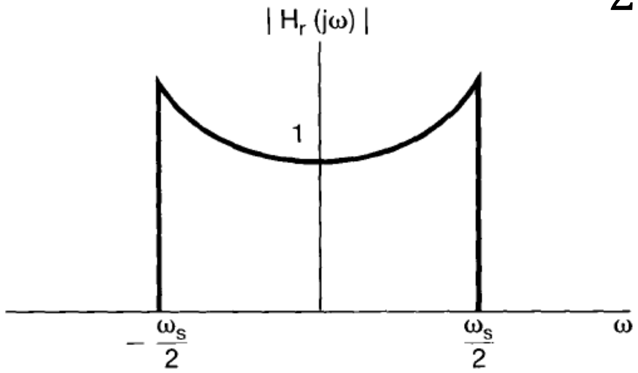

Let $H_0(j\omega)H_r(j\omega) = H(j\omega)$

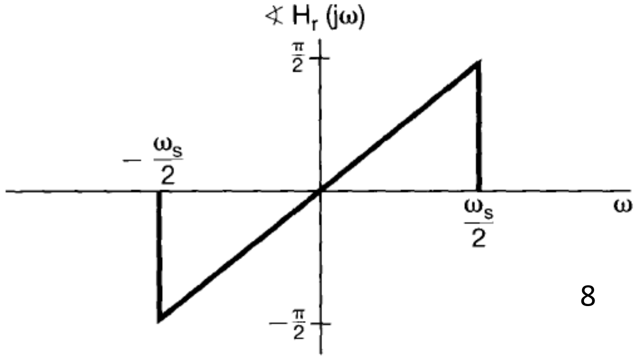
$$H_r(j\omega) = \begin{cases} e^{j\omega T/2} / \left[\dfrac{2\sin(\omega T/2)}{\omega}\right], & |\omega| \le \dfrac{\omega_s}{2} \\ 0, & |\omega| > \dfrac{\omega_s}{2} \end{cases}$$

$$H_0(j\omega) = e^{-j\omega T/2}\left[\frac{2\sin(\omega T/2)}{\omega}\right]$$



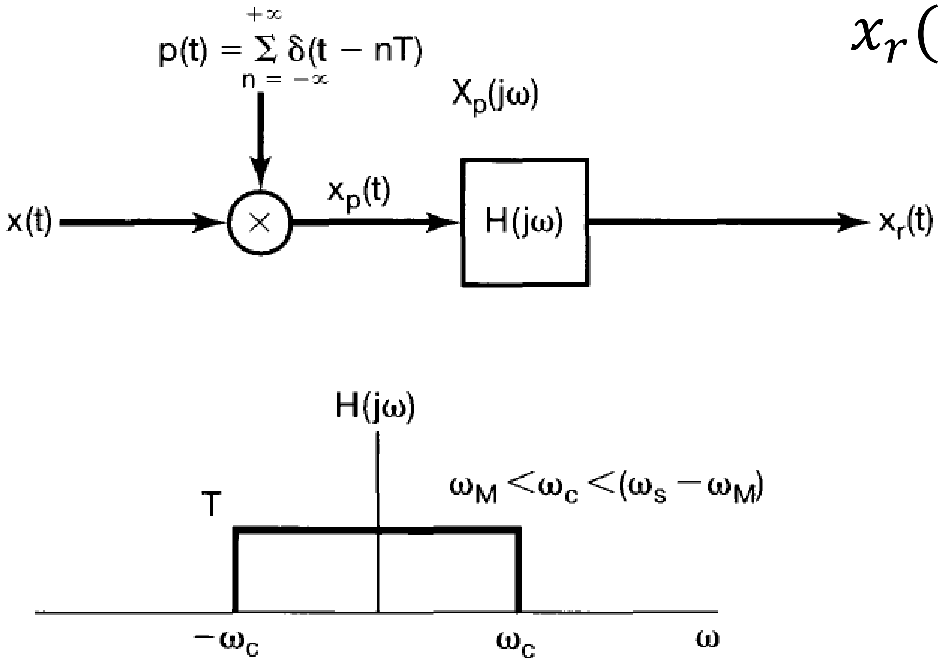



# Sampling (ch.7)

- ❑ Representation of a Continuous-Time Signal by Its Samples: The Sampling Theorem
- ❑ Reconstruction of a Signal from Its Samples Using Interpolation
- ❑ The Effect of Undersampling: Aliasing
- ❑ Discrete-Time Processing of Continuous-Time Signals
- ❑ Sampling of Discrete-Time Signals

# Reconstruction of a Signal Using Interpolation

***Band-limited interpolation:*** (ideal low-pass filter)




$$x_r(t) = x_p(t) * h(t) = \left[\sum_{n=-\infty}^{\infty} x(nT) \cdot \delta(t - nT)\right] * h(t)$$

$x_p(t)$

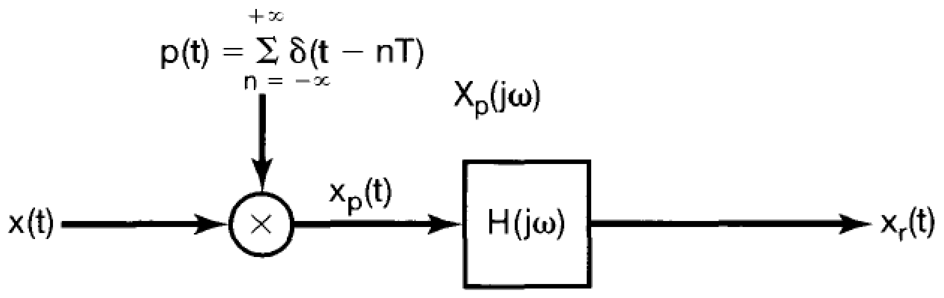
$$= \sum_{n=-\infty}^{\infty} x(nT)[\delta(t - nT) * h(t)]$$

$$x_r(t) = \sum_{n=-\infty}^{\infty} x(nT)h(t - nT)$$

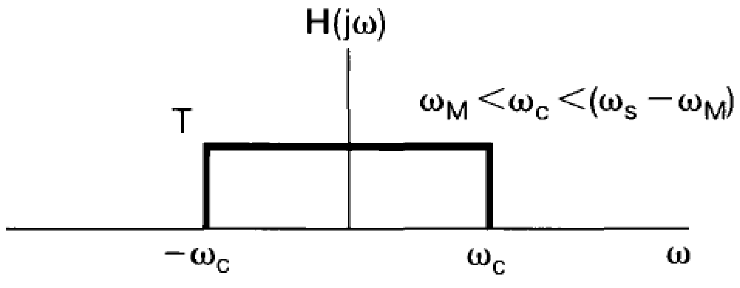
Shifted ($nT$) and weighted [$x(nT)$] sum of $h(t)$

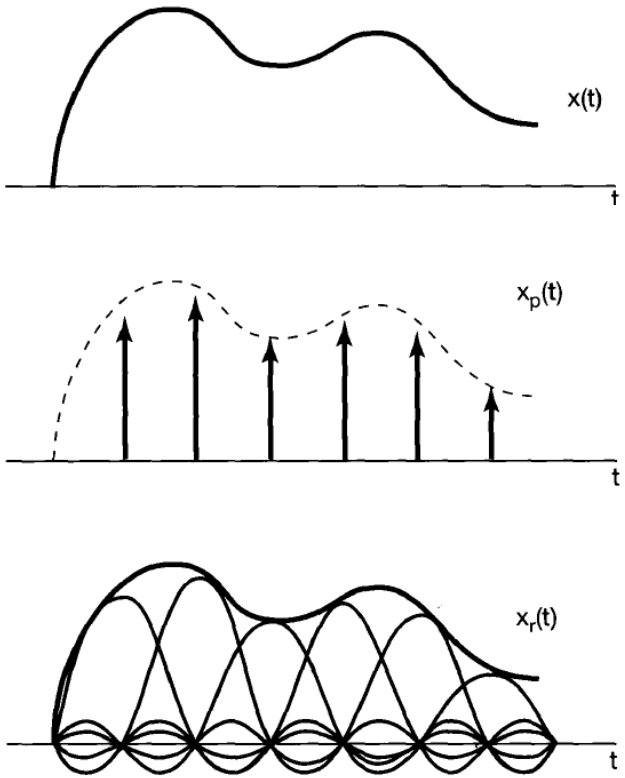
# Reconstruction of a Signal Using Interpolation

***Band-limited interpolation:*** (ideal low-pass filter)




$$x_r(t) = \sum_{n=-\infty}^{\infty} x(nT)\frac{T\omega_c}{\pi}\frac{\sin \omega_c (t - nT)}{\omega_c (t - nT)}$$

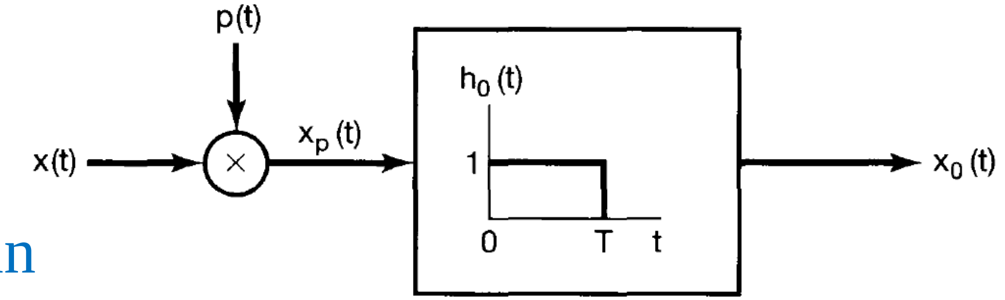



$$h(t) = \frac{T\omega_c}{\pi}\frac{\sin \omega_c t}{\omega_c t}$$

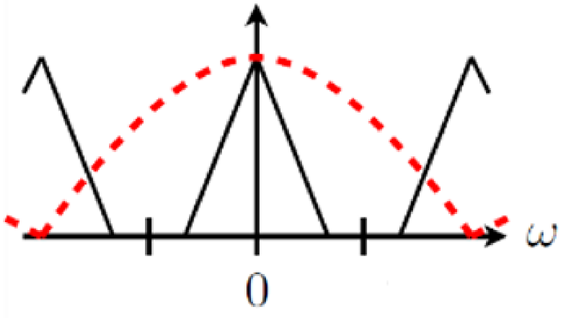
# Reconstruction of a Signal Using Interpolation

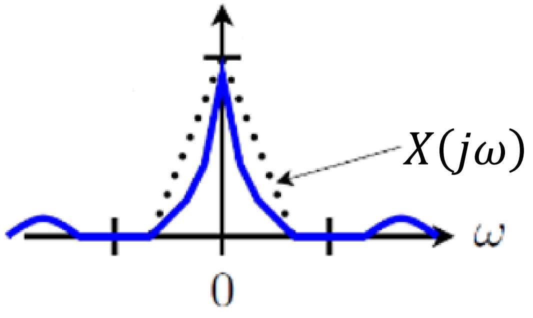
## ***Zero-order hold***



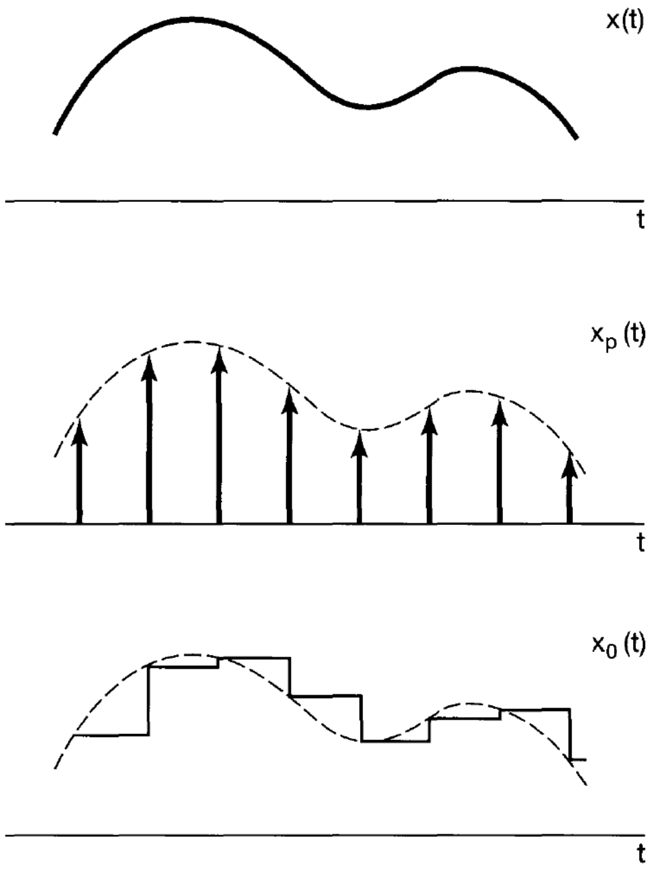

- Frequency domain

$$H_0(j\omega) = e^{-j\omega T/2}\left[\frac{2\sin(\omega T/2)}{\omega}\right]$$



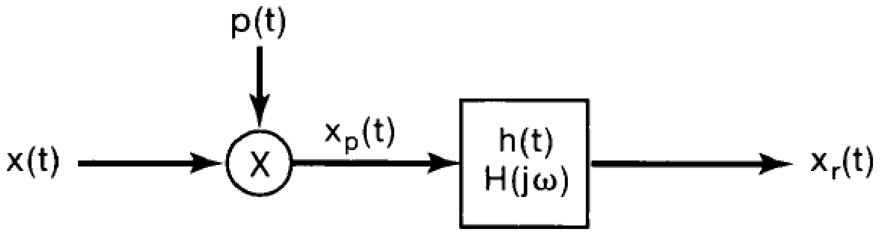

- Time domain



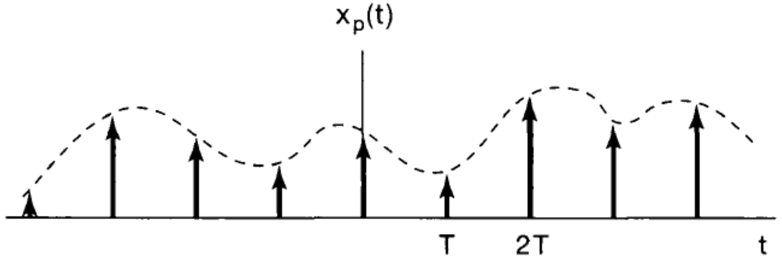

Shifted and weighted sum of $h(t)$

# Reconstruction of a Signal Using Interpolation

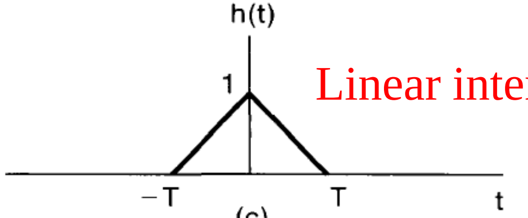
***First-order hold:*** Impulse-train sampling + an LTI system with a tri angular impulse response




- Time domain

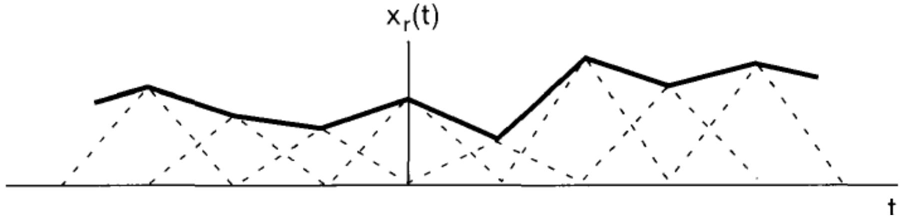



(b)




Linear interpolation

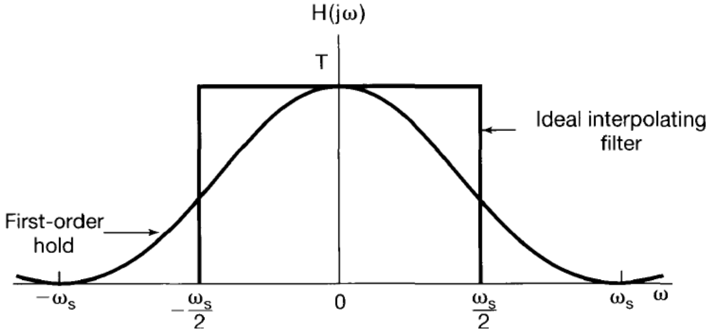
(c)




Shifted and weighted sum of $h(t)$

- Frequency domain

$$H(j\omega) = \frac{1}{T}\left[\frac{\sin(\omega T/2)}{\omega/2}\right]^2$$



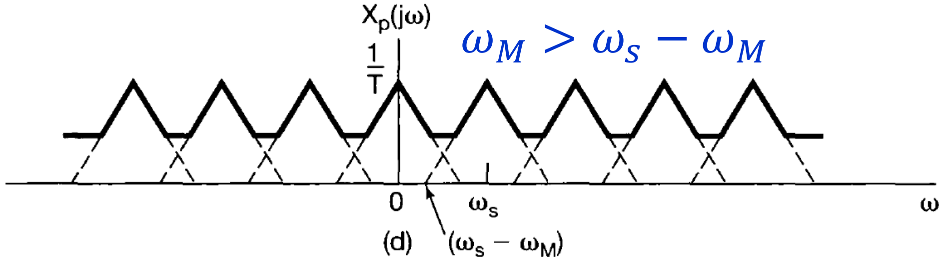
# Sampling (ch.7)

- ❑ Representation of a Continuous-Time Signal by Its Samples: The Sampling Theorem
- ❑ Reconstruction of a Signal from Its Samples Using Interpolation
- ❑ The Effect of Undersampling: Aliasing
- ❑ Discrete-Time Processing of Continuous-Time Signals
- ❑ Sampling of Discrete-Time Signals

# Aliasing

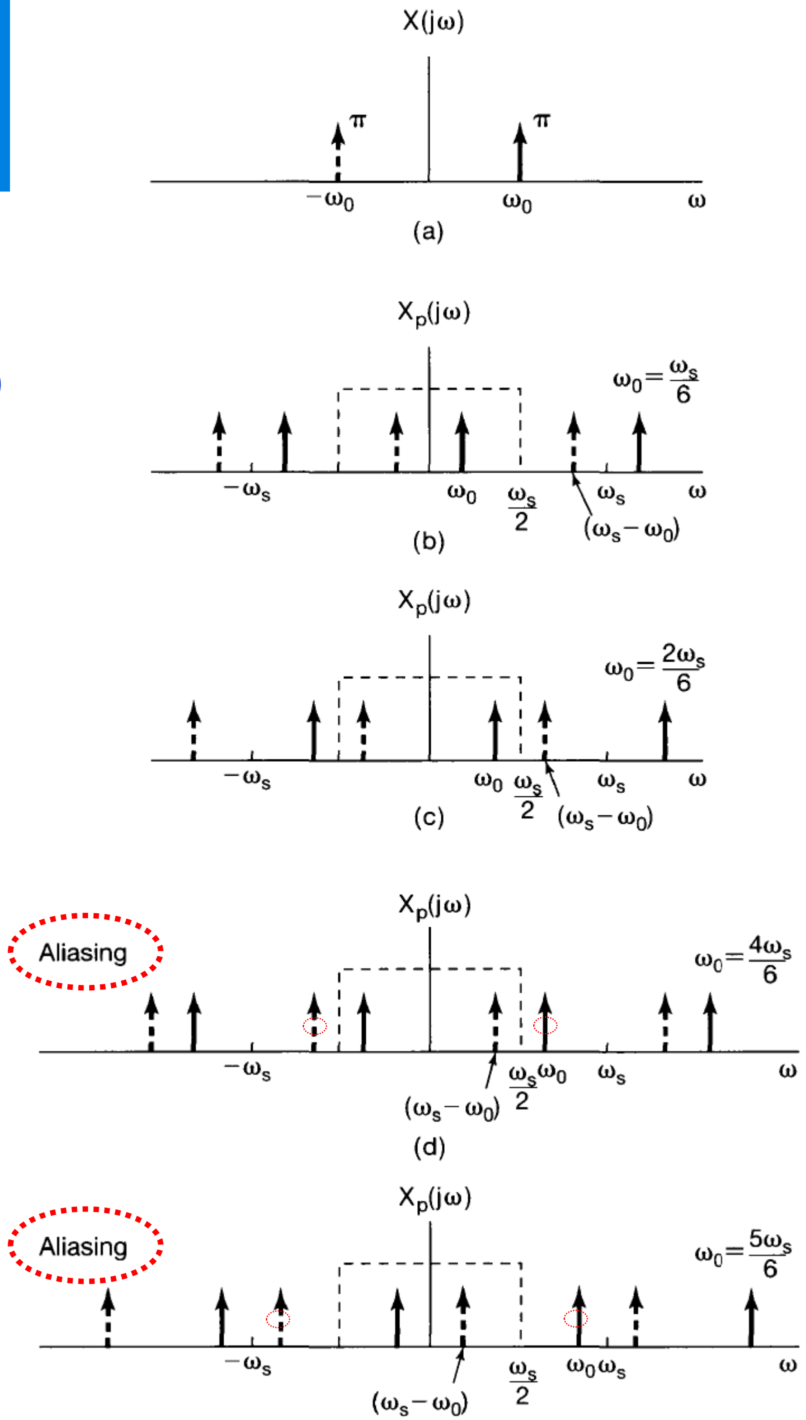
## *Aliasing*

- When $\omega_s < 2\omega_M$, the individual spectrums overlap




- Consider original signal is $x(t) = \cos\omega_0 t$, with different $\omega_0$ but sampled at same $\omega_s$

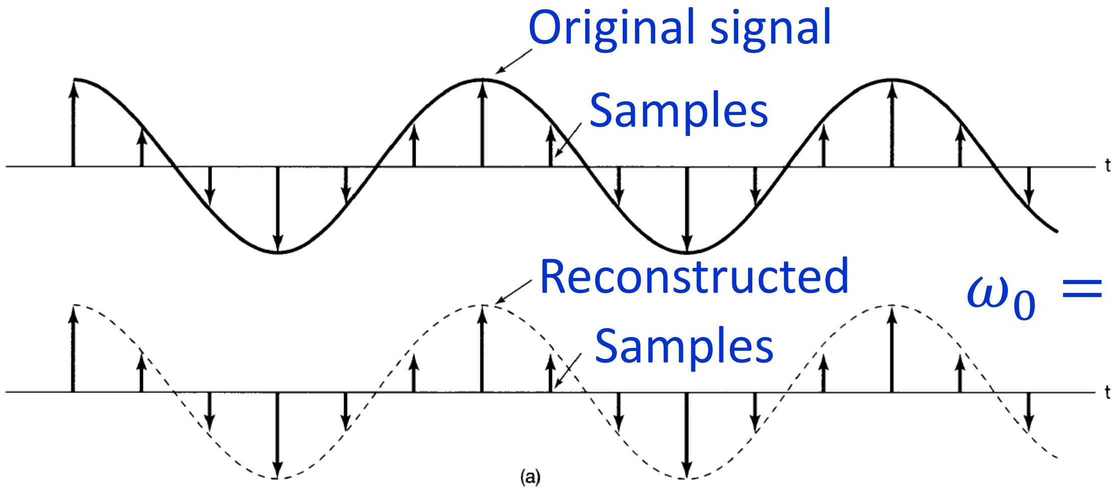
  - When aliasing occurs, the original frequency $\omega_0$ takes on the identity of lower frequency $(\omega_s - \omega_0)$.



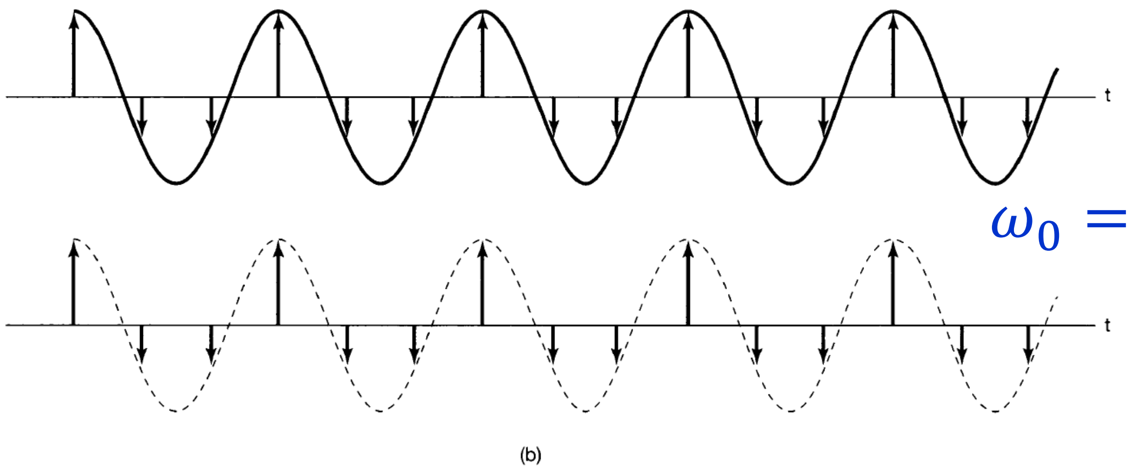
# Aliasing

## *Aliasing*

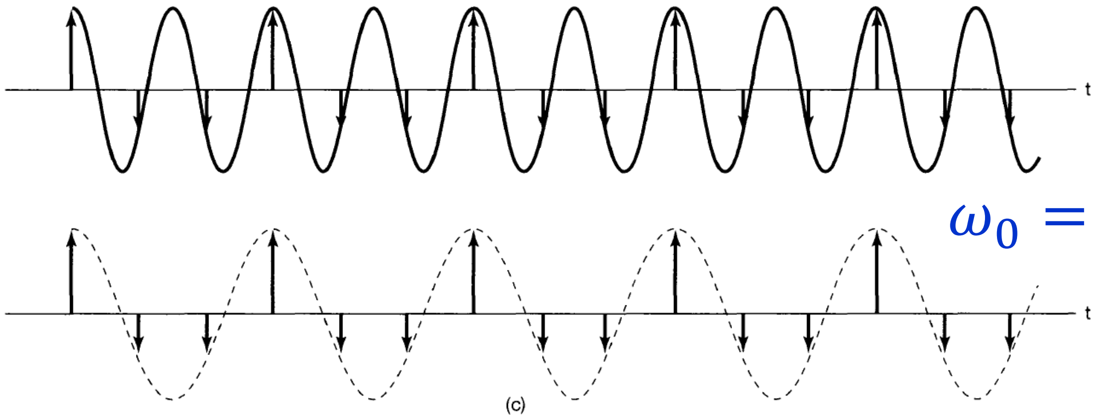
$$x(t) = \cos \omega_0 t \quad \text{Time domain}$$



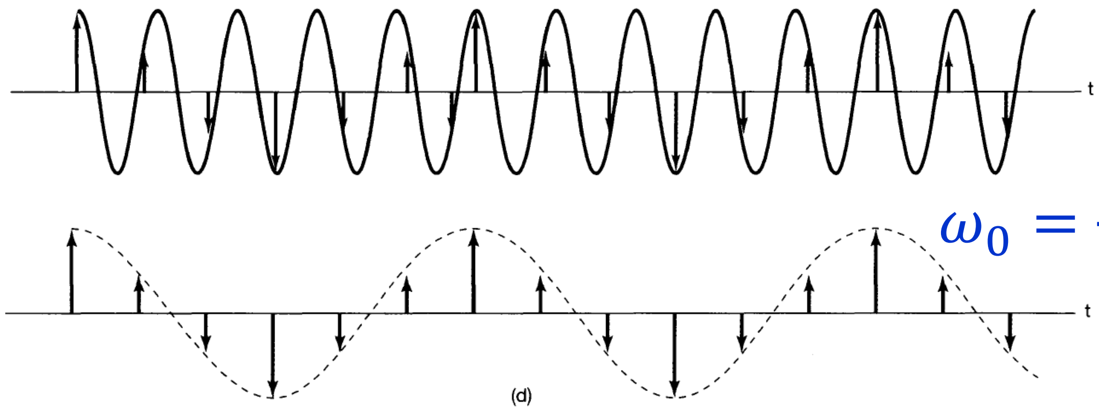

$$\omega_0 = \frac{\omega_s}{6}$$




$$\omega_0 = \frac{2\omega_s}{6}$$




$$\omega_0 = \frac{4\omega_s}{6}$$




$$\omega_0 = \frac{5\omega_s}{6}$$

# Aliasing

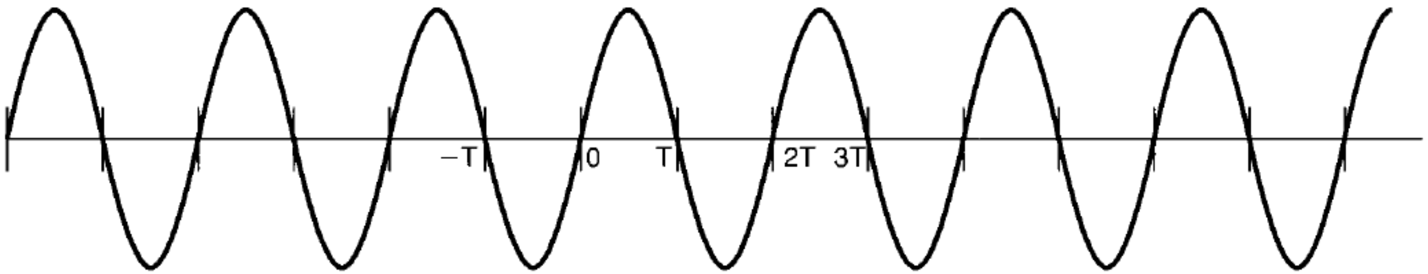
## Aliasing

- $\omega_s = 2\omega_M$ is not sufficient to avoid aliasing
  - Consider a signal $x(t) = \cos(\omega_0 t + \emptyset)$ is sampled using impulse sampling with $\omega_s = 2\omega_0$
  - The reconstructed signal using ideal low-pass filter is

$$x_r(t) = \cos(\emptyset)\cos(\frac{\omega_s}{2}t)$$

$x_r(t) = x(t)$ only if $\emptyset = 2k\pi$

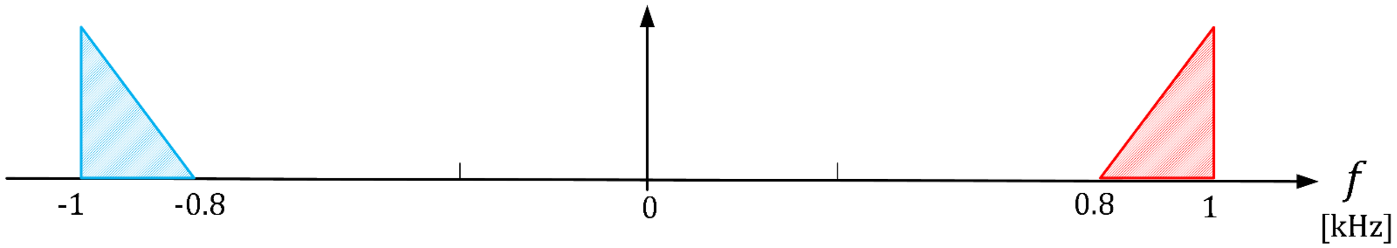
  - Particularly, if $\emptyset = -\pi/2$, then $x(t) = \sin\omega_0 t$ and $x_r(t) = 0$



# Aliasing

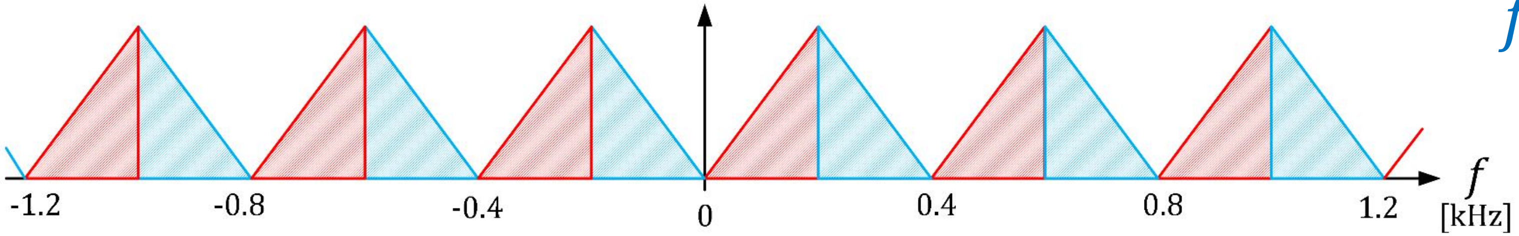
## Aliasing

- For signal with $f_c > B/2$, where $f_c = (f_h + f_l)/2$ and $B = f_h - f_l$




$f_l = 800\text{ Hz}, f_h = 1000\text{ Hz}$

Determine the lowest $f_s$ with no aliasing

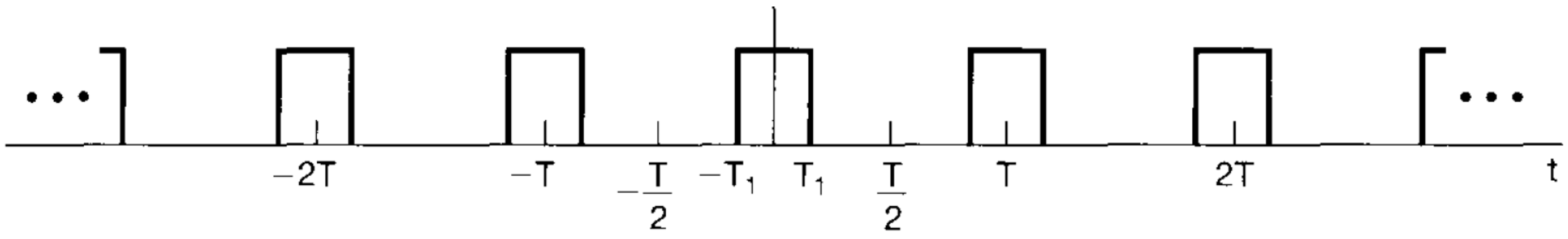


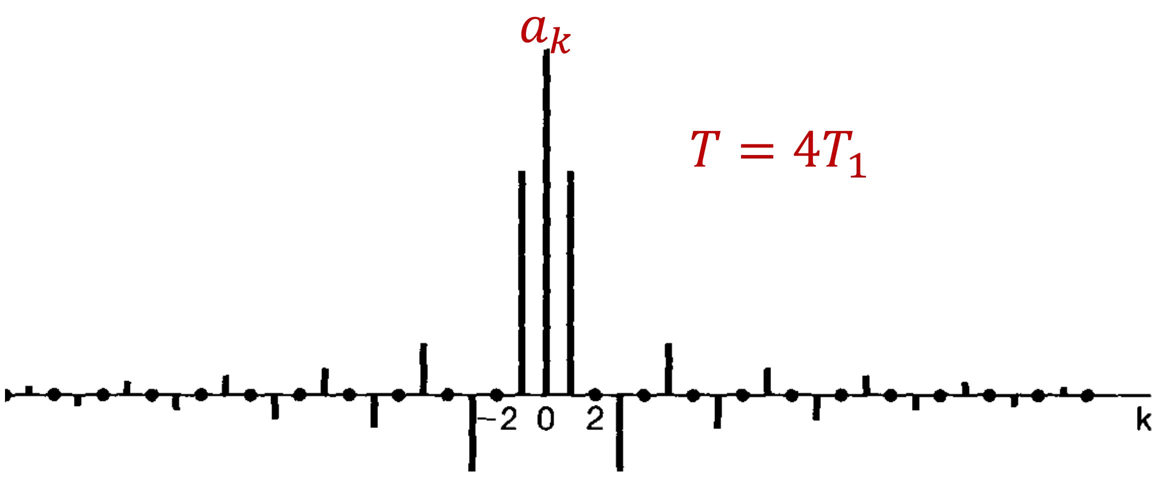

$f_s = 400$ Hz

$Q$: What about $f_l = 850\text{ Hz}$?

## *Aliasing*

- For harmonic related signal, e.g., a square wave



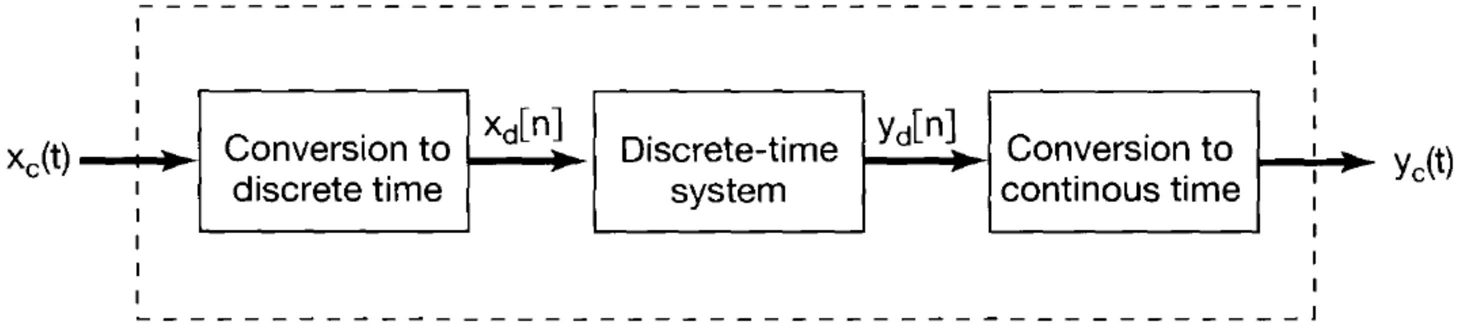

- $\omega_s > 2K\omega_0$, with $K$ the kth harmonics you want to include
- Low-pass filtering before sampling

# Sampling (ch.7)

- ❑ Representation of a Continuous-Time Signal by Its Samples: The Sampling Theorem
- ❑ Reconstruction of a Signal from Its Samples Using Interpolation
- ❑ The Effect of Undersampling: Aliasing
- ❑ Discrete-Time Processing of Continuous-Time Signals
- ❑ Sampling of Discrete-Time signals

# Discrete-Time Processing of Continuous-Time Signals

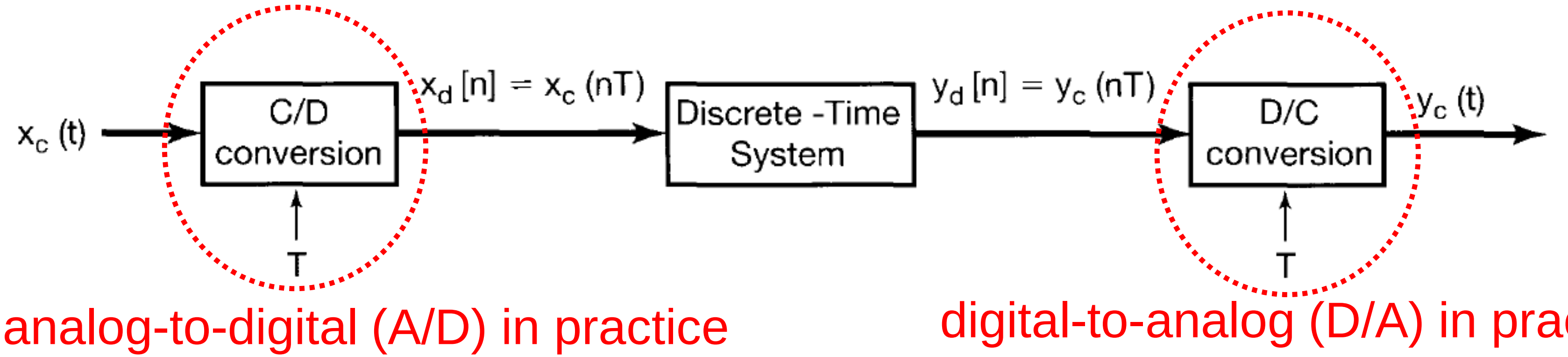
## ***General scheme***




$$x_d[n] = x_c(nT)$$

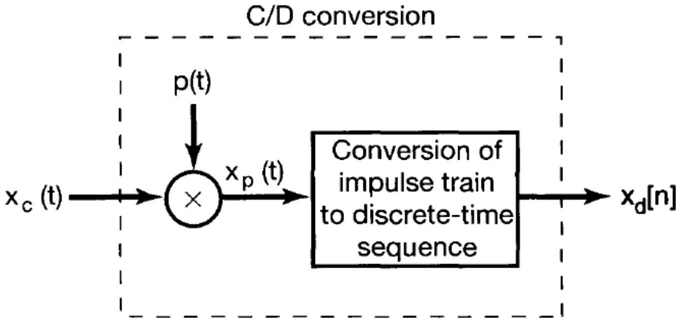
$$y_d[n] = y_c(nT)$$

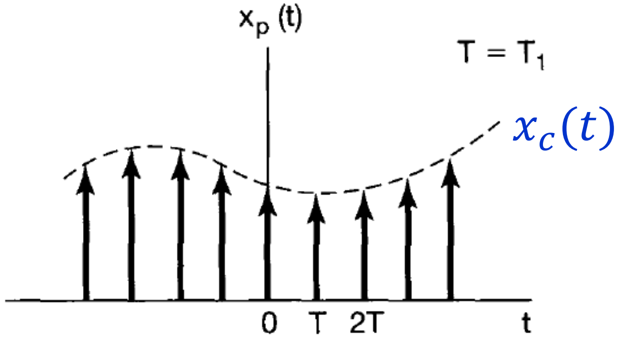
- C/D: continuous-to-discrete-time conversion
- D/C: discrete-to-continuous-time conversion

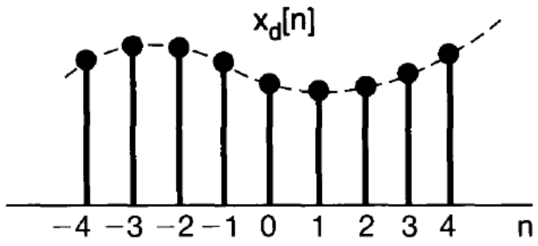


# Discrete-Time Processing of Continuous-Time Signals

## ***C/D conversion*** Time domain



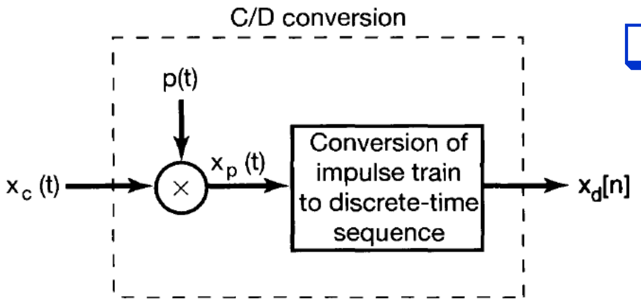

$$x_p(t) = \sum_{n=-\infty}^{\infty} x_c(nT) \cdot \delta(t - nT)$$

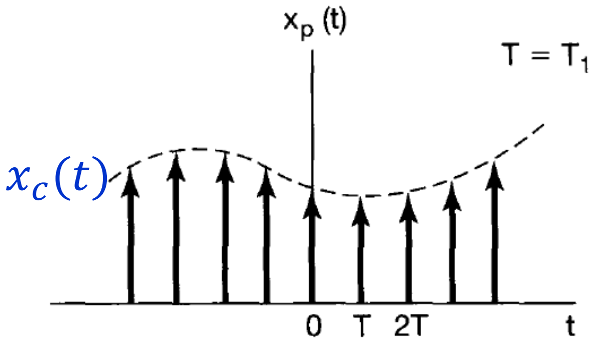
$$x_d[n] = x_c(nT)$$

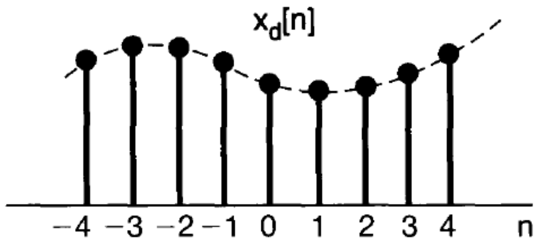
# Discrete-Time Processing of Continuous-Time Signals

## C/D conversion

Frequency domain: $\omega$ for continuous time and $\Omega$ for discrete time




- Spectrum of $x_d[n]$

$$X_d(e^{j\Omega}) = \sum_{n=-\infty}^{\infty} x_d[n]e^{-jn\Omega} = \sum_{n=-\infty}^{\infty} x_c(nT)e^{-jn\Omega}$$

- Spectrum of $x_p(t)$

$$x_p(t) = \sum_{n=-\infty}^{\infty} x_c(nT)\cdot\delta(t-nT) \implies X_p(j\omega) = \sum_{n=-\infty}^{\infty} x_c(nT)\cdot e^{-j\omega nT}$$

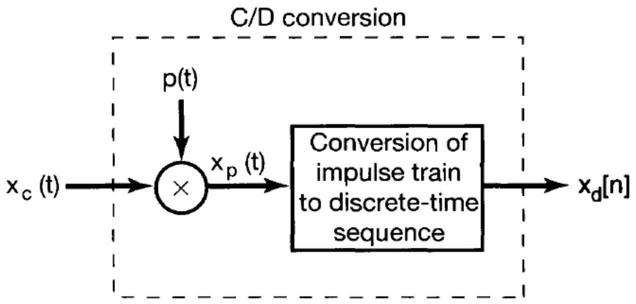
- If $\omega = \Omega/T$, $X_d(e^{j\Omega}) = X_p(j\Omega/T)$

- The spectrum of $x_d[n]$ can be obtained from $X_p(j\omega)$ by replacing $\omega$ with $\Omega/T$.

# Discrete-Time Processing of Continuous-Time Signals

## C/D conversion

Frequency domain: $\omega$ for continuous time and $\Omega$ for discrete time

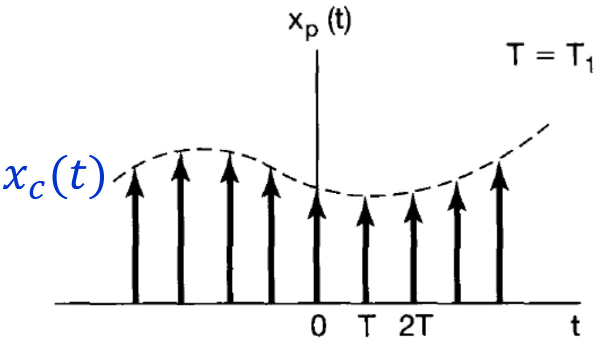


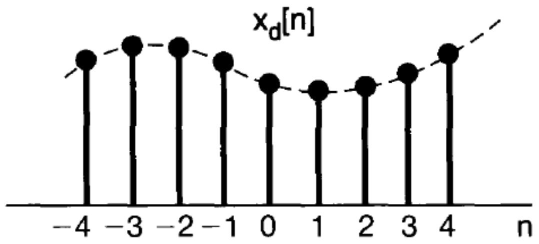

- Recall: $X_p(j\omega) = \frac{1}{T}\sum_{K=-\infty}^{\infty} X\big(j(\omega - k\cdot\omega_s)\big)$

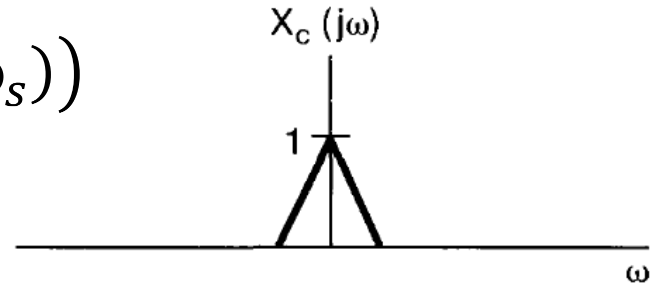
$$\therefore X_d\left(e^{j\Omega}\right) = \frac{1}{T}\sum_{K=-\infty}^{\infty} X(j(\Omega - 2k\pi)/T)$$

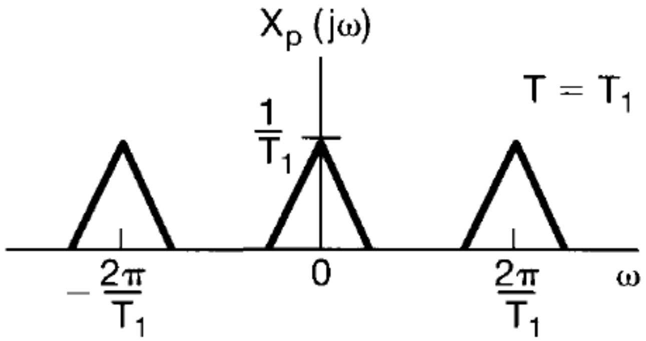
- $X_d\left(e^{j\Omega}\right)$ is a frequency-scaled version of $X_p(j\omega)$, and is periodic with period of $2\pi$

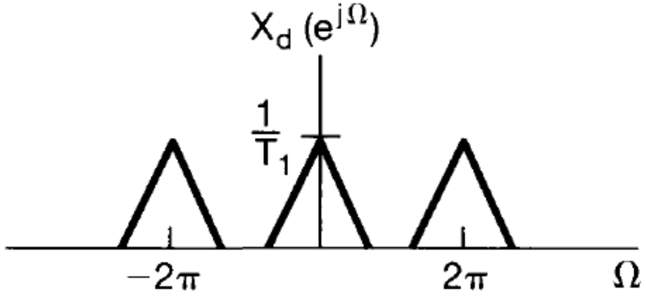
- Informally
  - $t$ to $n$: time scaling by 1/ $T$
  - $\omega$ to $\Omega$: frequency scaling by $T$



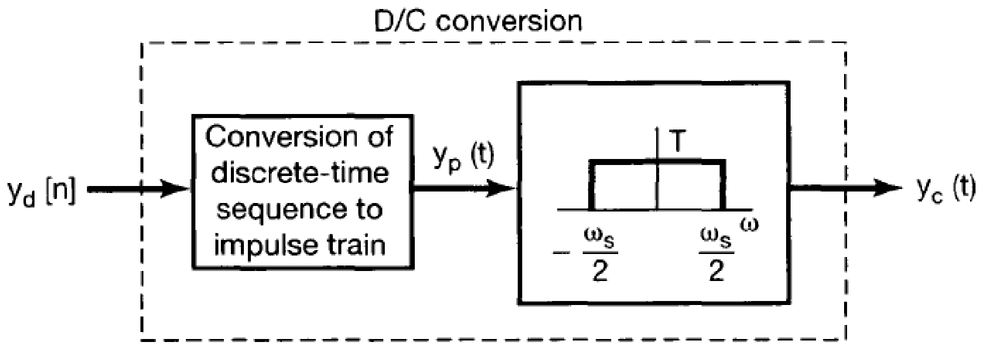



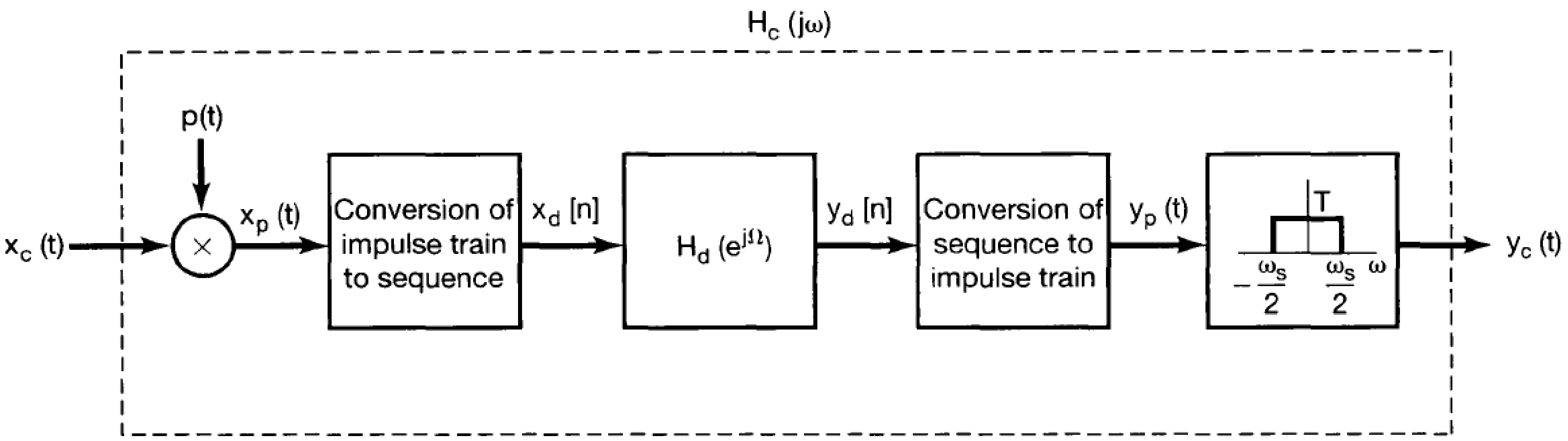
## D/C conversion




- ❑ $Y_d(e^{j\Omega})$: Spectrum of $y_d[n]$
- ❑ $Y_p(j\omega)$: Spectrum of $y_p(t)$
- ❑ $Y_p(j\omega)$ can be obtained from $Y_d(e^{j\Omega})$ by replacing $\Omega$ with $\omega T$.

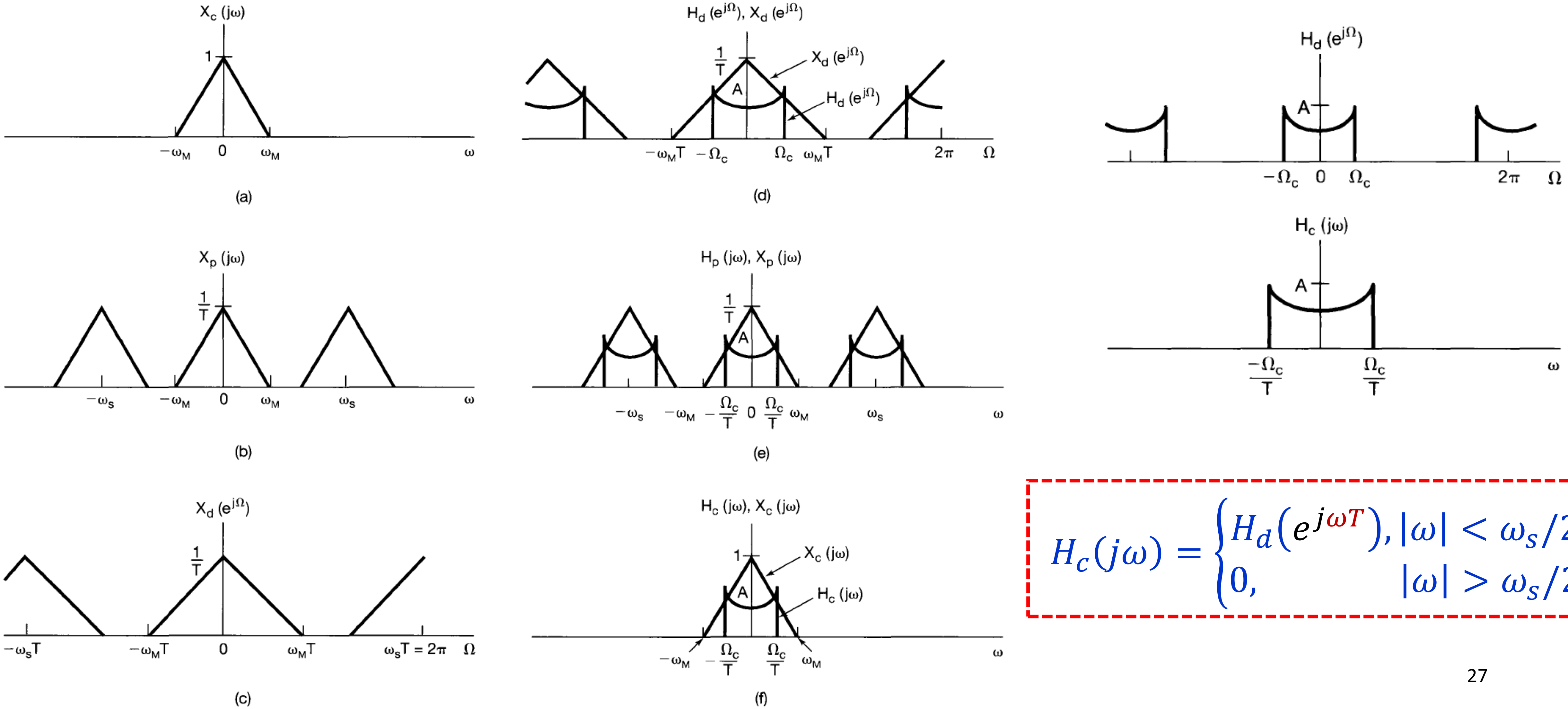
# Discrete-Time Processing of Continuous-Time Signals

## ***Overall system***



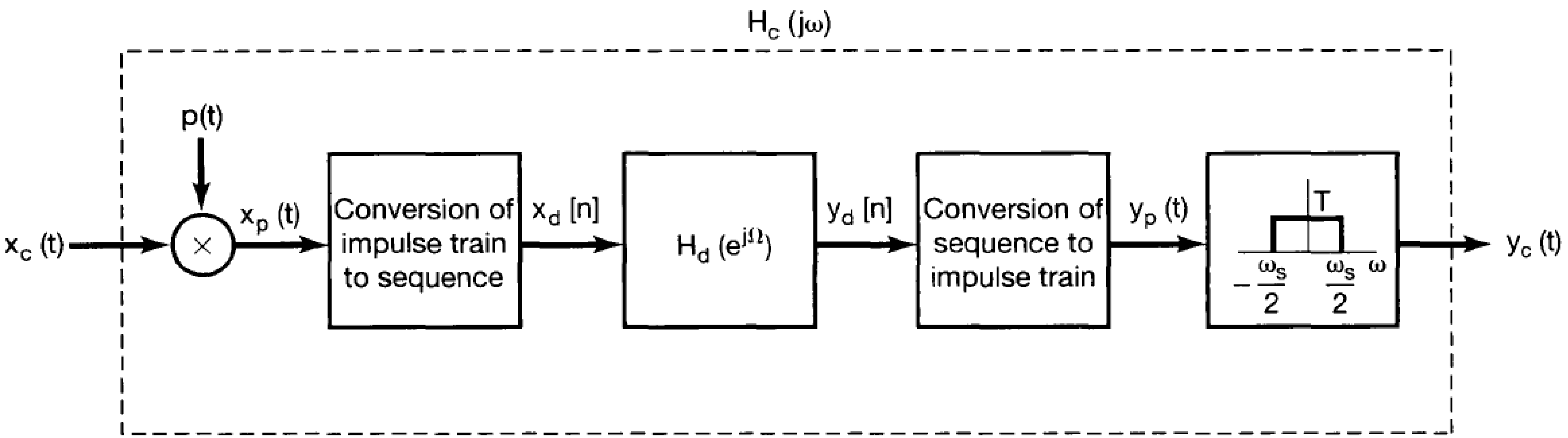

- ❑ $x_c(t)$: input
- ❑ $y_c(t)$: output
- ❑ The overall system is equivalent to a continuous-time system with frequency response $H_c(j\omega)$
- ❑ $H_c(j\omega) =?$

## Overall system

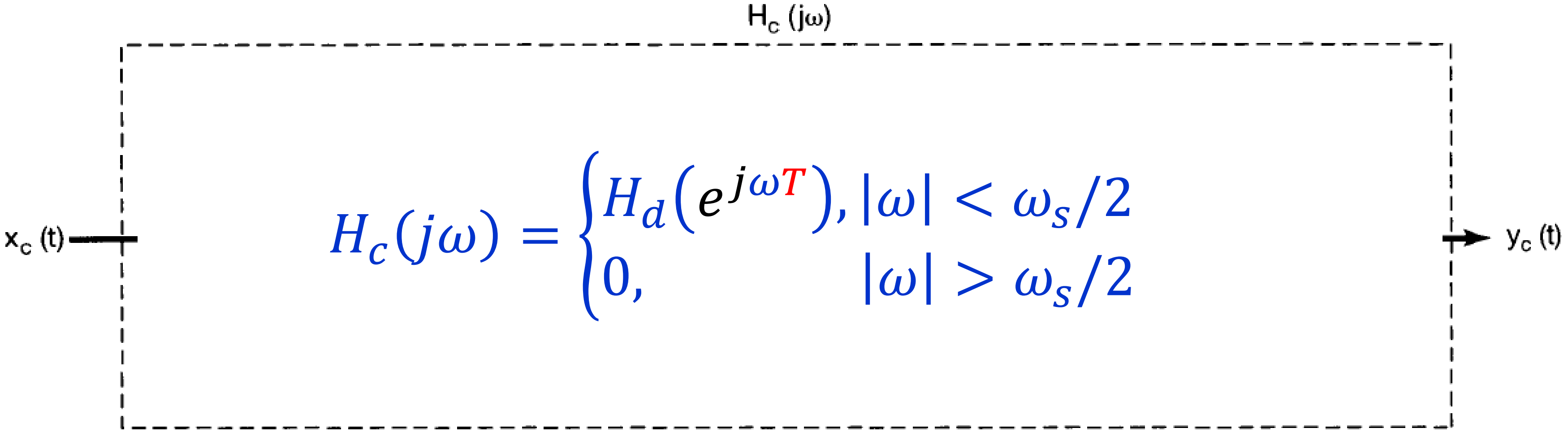



$$H_c(j\omega) = \begin{cases} H_d(e^{j\omega T}), & |\omega| < \omega_s/2 \\ 0, & |\omega| > \omega_s/2 \end{cases}$$

## ***Overall system***




$\Updownarrow$ $\Omega = \omega T$




$$H_c(j\omega) = \begin{cases} H_d\left(e^{j\omega T}\right), & |\omega| < \omega_s/2 \\ 0, & |\omega| > \omega_s/2 \end{cases}$$

# Discrete-Time Processing of Continuous-Time Signals

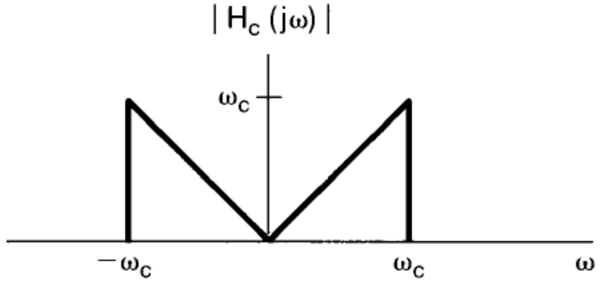
## ***Digital differentiator: frequency response***

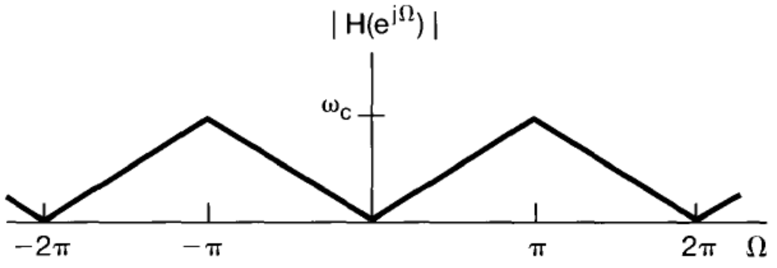
- ❑ Band-limited CT differentiator ⟺ ❑ Corresponding DT differentiator

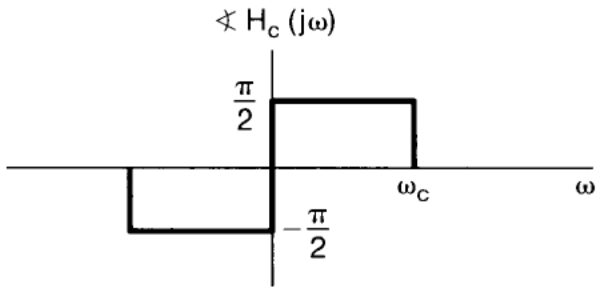
$$H_c(j\omega) = \begin{cases} j\omega, & |\omega| < \omega_c \\ 0, & |\omega| > \omega_c \end{cases}$$

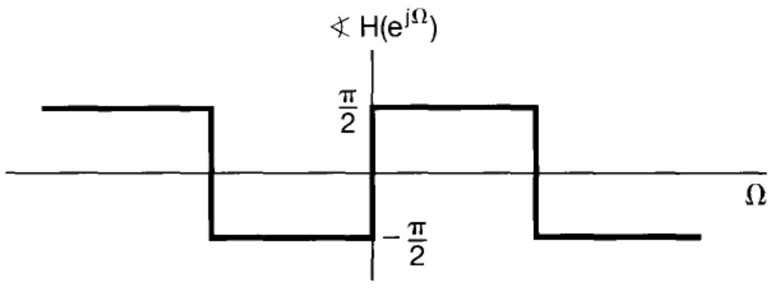
$$H_d(e^{j\Omega}) = j\frac{\Omega}{T}, |\Omega| < \pi$$

$\omega_c = \omega_s/2$



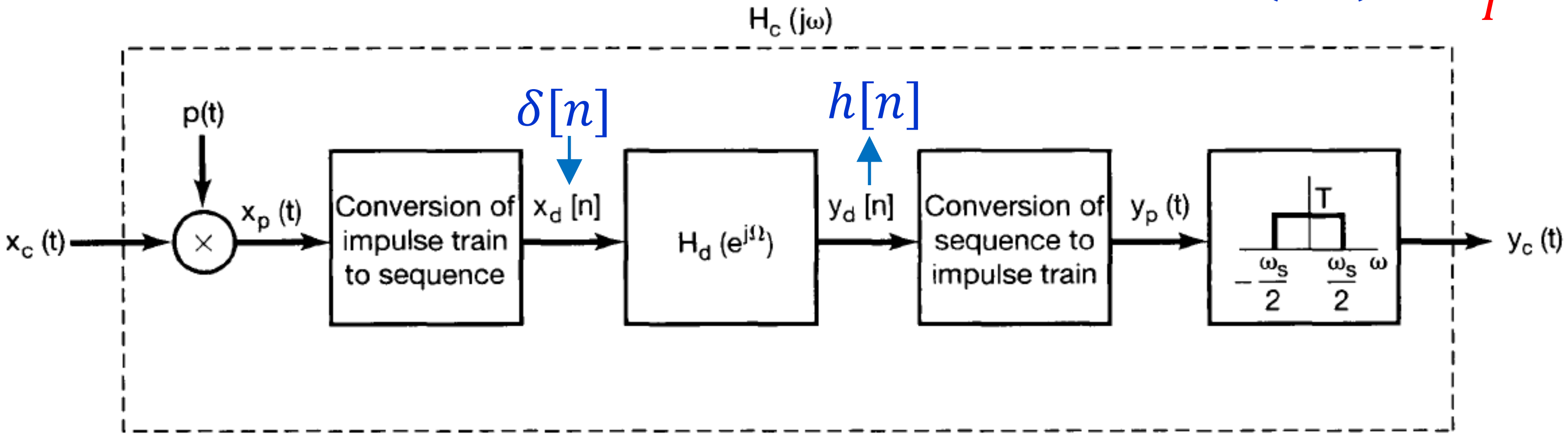



## Digital differentiator: impulse response

$$H_d(e^{j\Omega}) = j\frac{\Omega}{T}, |\Omega| < \pi$$




- **Assume** $x_c(t) = \dfrac{\sin(\pi t/T)}{\pi t}$ $\Rightarrow x_d[n] = x_c(nT) = \frac{1}{T}\delta[n]$

- $y_d[n] = y_c(nT)$ $\qquad y_c(t) = \dfrac{d}{dt}x_c(t) = \dfrac{\cos(\pi t/T)}{Tt} - \dfrac{\sin(\pi t/T)}{\pi t^2}$

- $y_d[n] = \begin{cases} \dfrac{(-1)^n}{nT^2}, n \neq 0 \\ 0, \ n = 0 \end{cases}$ $\Rightarrow$ $\therefore h_d[n] = \begin{cases} \dfrac{(-1)^n}{nT}, n \neq 0 \\ 0, \qquad n = 0 \end{cases}$

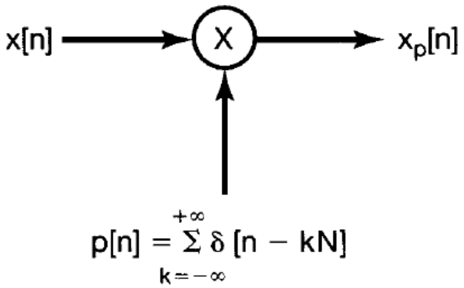
# Sampling (ch.7)

- ❑ Representation of a Continuous-Time Signal by Its Samples: The Sampling Theorem
- ❑ Reconstruction of a Signal from Its Samples Using Interpolation
- ❑ The Effect of Undersampling: Aliasing
- ❑ Discrete-Time Processing of Continuous-Time Signals
- ❑ Sampling of Discrete-Time signals

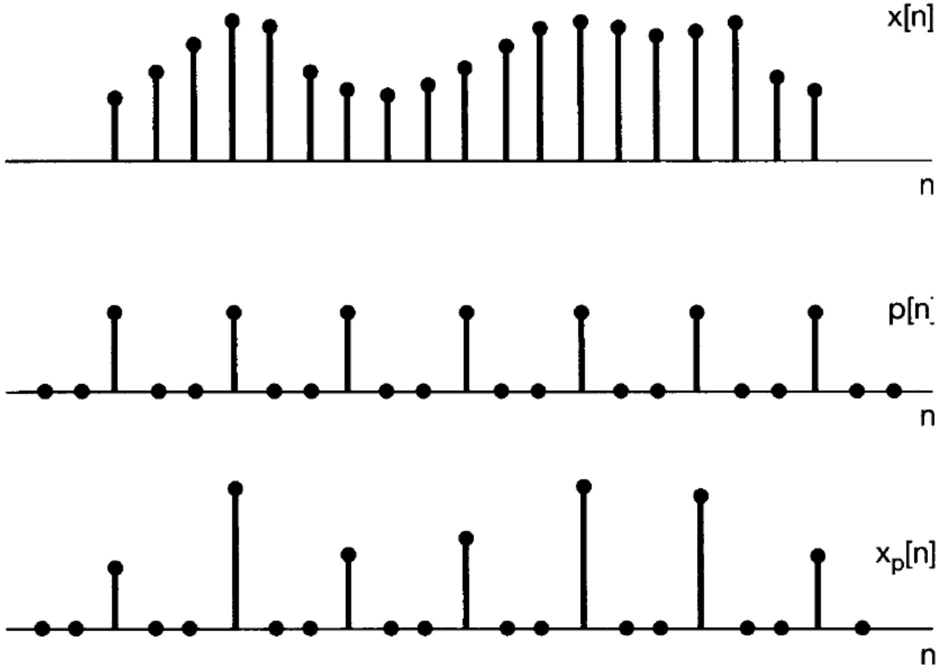
# Sampling of Discrete-Time Signals

## *Impulse train sampling*

Time domain




N: sampling period




$$x_p[n] = x[n]p[n] = \sum_{n=-\infty}^{\infty} x[kN]\delta[n - kN]$$

$$= \begin{cases} x[n], \text{if } n \text{ is an integer multiple of N} \\ 0, \qquad\qquad\qquad\qquad\qquad \text{otherwise} \end{cases}$$

# Sampling of Discrete-Time Signals

## ***Impulse train sampling***

Frequency domain

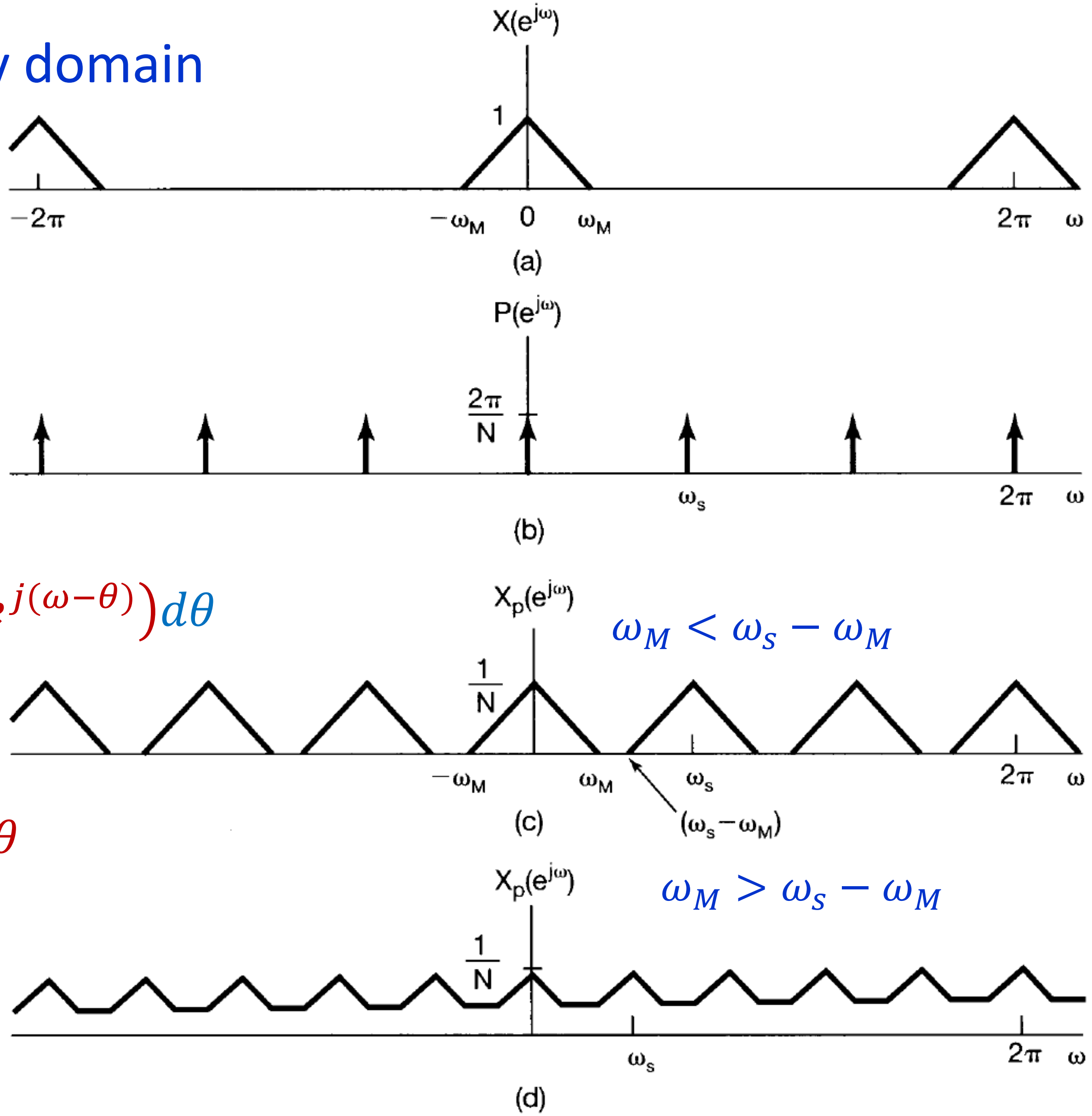
$$P(e^{j\omega}) = \frac{2\pi}{N}\sum_{K=-\infty}^{\infty}\delta(\omega - k\omega_s) \quad \omega_s = \frac{2\pi}{N}$$

$$X_p(e^{j\omega}) = \frac{1}{2\pi}\int_{2\pi} P(e^{j\theta})X(e^{j(\omega-\theta)})d\theta$$

$$= \frac{1}{2\pi}\cdot\frac{2\pi}{N}\int_{2\pi}\left[\sum_{K=-\infty}^{\infty}\delta(\theta - k\omega_s)\right]X(e^{j(\omega-\theta)})d\theta$$

$$= \frac{1}{N}\sum_{K=0}^{N-1}\int_{2\pi}\delta(\theta - k\omega_s)X(e^{j(\omega-\theta)})d\theta$$

$$X_p(e^{j\omega}) = \frac{1}{N}\sum_{K=0}^{N-1}X(e^{j(\omega-k\cdot\omega_s)})$$



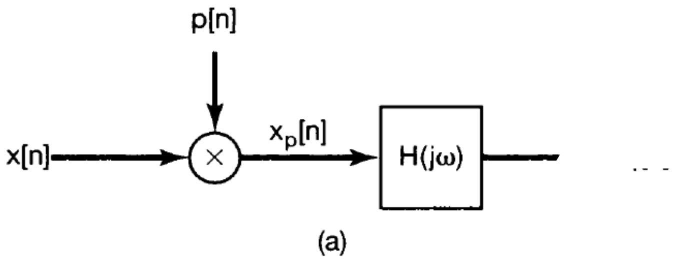

$\omega_M < \omega_s - \omega_M$

$\omega_M > \omega_s - \omega_M$

# Sampling of Discrete-Time Signals

***Impulse train sampling***

Reconstruction of x[n]




Time domain

$$x_r[n] = x_p[n] * h[n]$$

$$= \left[\sum_{k=-\infty}^{\infty} x[kN]\cdot\delta[n-kN]\right] * h[n]$$

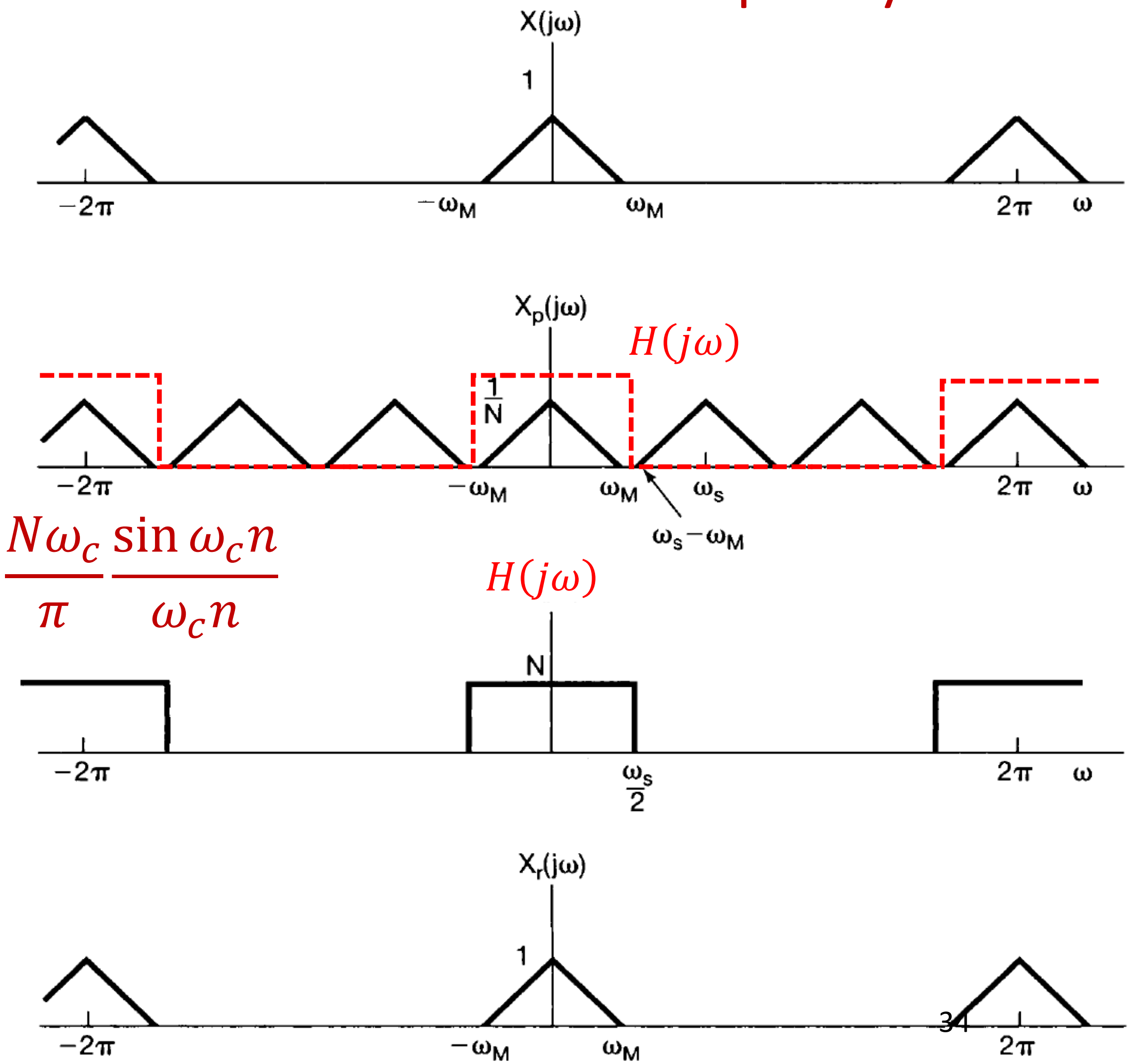
$$= \sum_{k=-\infty}^{\infty} x[kN][\delta[n-kN] * h[n]]$$

$$= \sum_{k=-\infty}^{\infty} x[kN]h[n-kN]$$

$$x_r[n] = \sum_{k=-\infty}^{\infty} x[kN]\frac{N\omega_c}{\pi}\frac{\sin\omega_c(n-kN)}{\omega_c(n-kN)}$$

Frequency domain

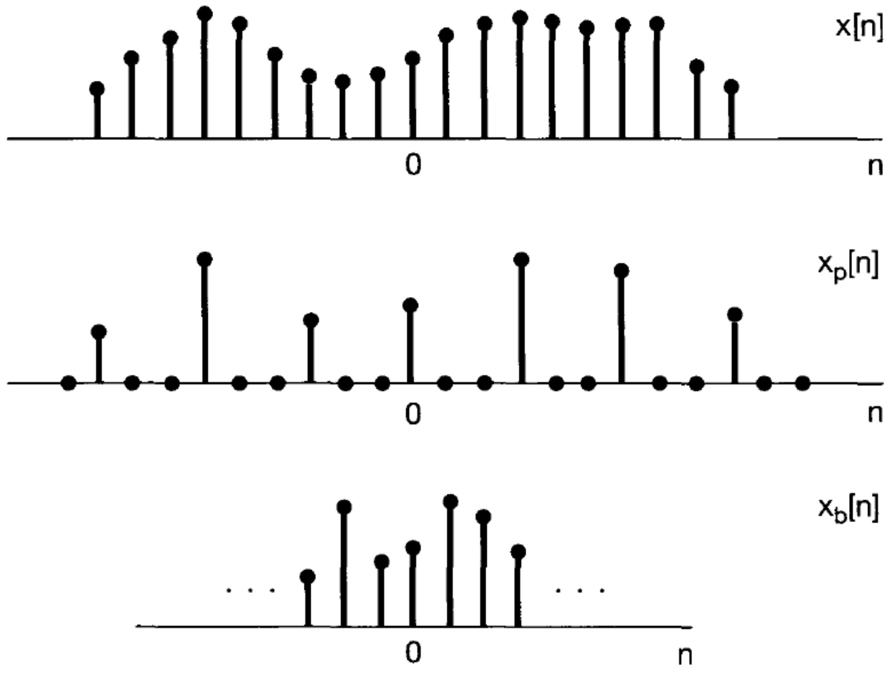
$$h[n] = \frac{N\omega_c}{\pi}\frac{\sin\omega_c n}{\omega_c n}$$



# Sampling of Discrete-Time Signals

## ***Decimation (sample rate decrease, SRD)***

Time domain




$$x_b[n] = x_p[nN]$$

$$x_b[n] = x[nN]$$

Frequency domain

$$X_b(e^{j\omega}) = \sum_{K=-\infty}^{\infty} x_b[k]e^{-j\omega k}$$

$$X_b(e^{j\omega}) = \sum_{K=-\infty}^{\infty} x_p[kN]e^{-j\omega k}$$

$n = kN$

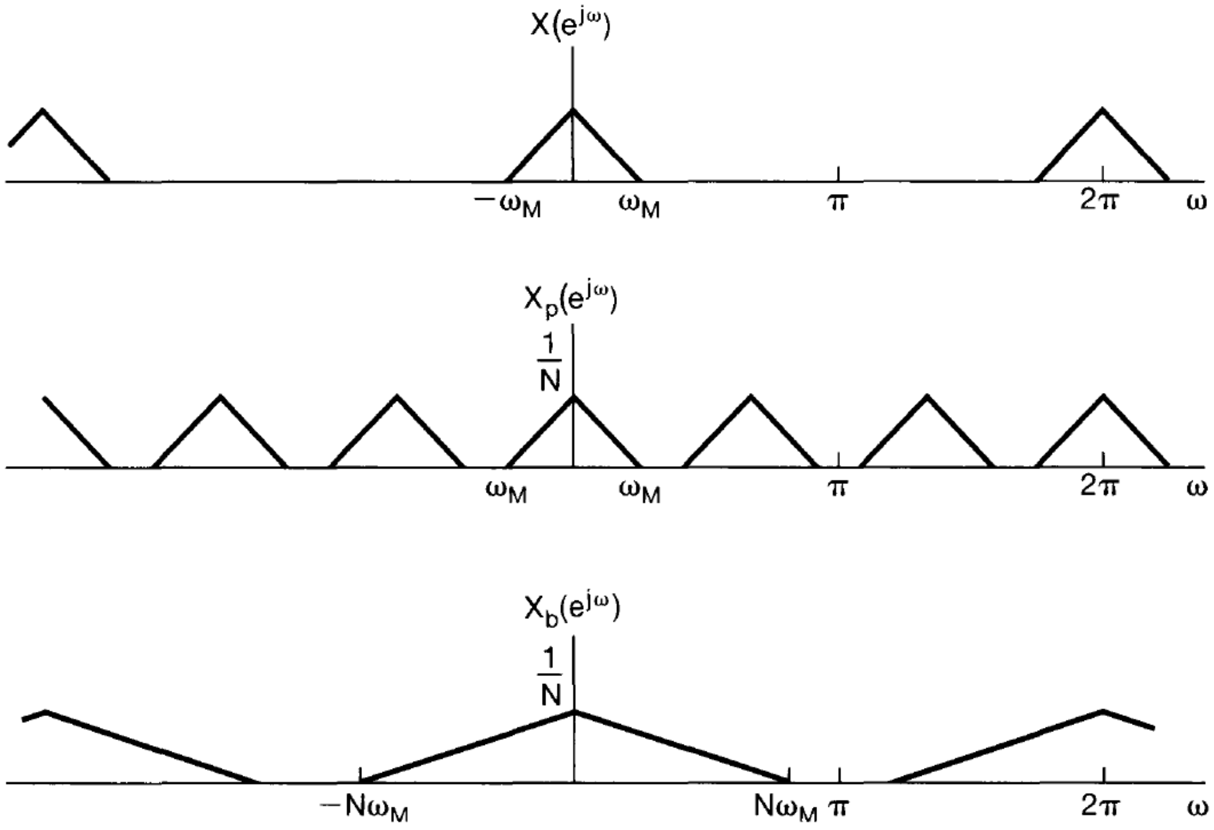
$$X_b(e^{j\omega}) = \sum_{\substack{n=\text{integer} \\ \text{number of N}}} x_p[n]e^{-j\omega n/N}$$

$$X_b(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x_p[n]e^{-j\omega n/N}$$

$$X_b(e^{j\omega}) = X_p(e^{j\omega/N})$$

# Sampling of Discrete-Time Signals

## ***Decimation***

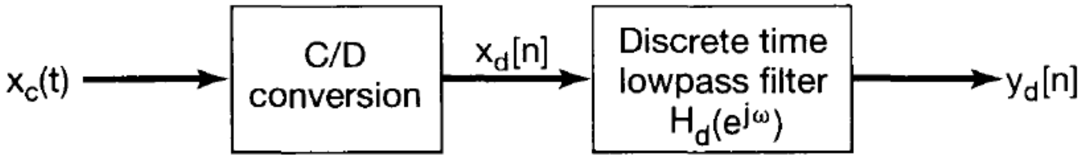



$$x_b[n] = x_p[nN]$$

$$X_b(e^{j\omega}) = X_p(e^{j\omega/N})$$

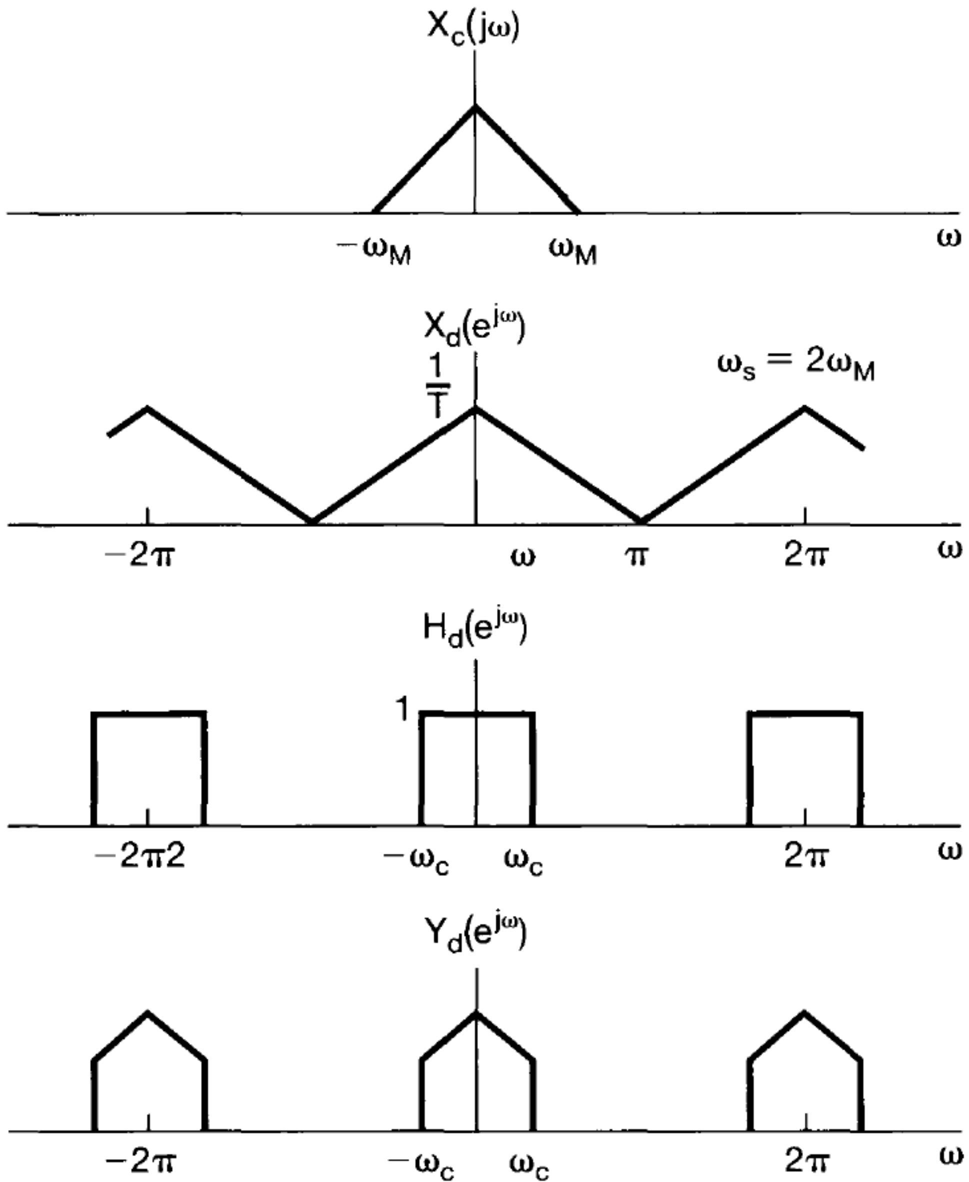
Down-sampling if $N\omega_M > \pi$

# Sampling of Discrete-Time



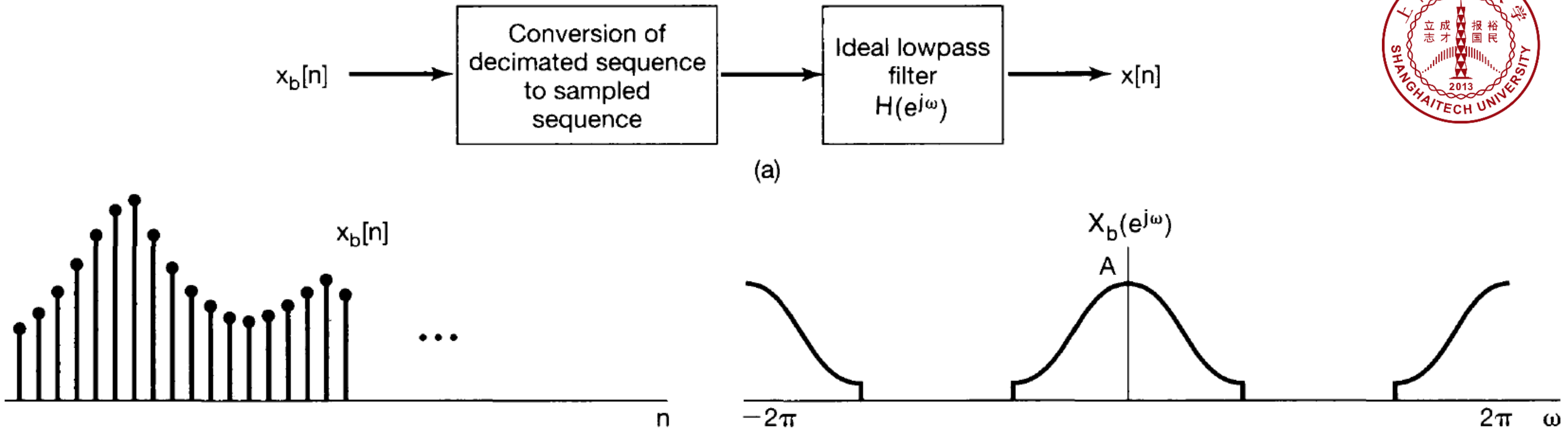

***Decimation***

- Prevent aliasing by LPF in front of SRD ⟹ Decimator

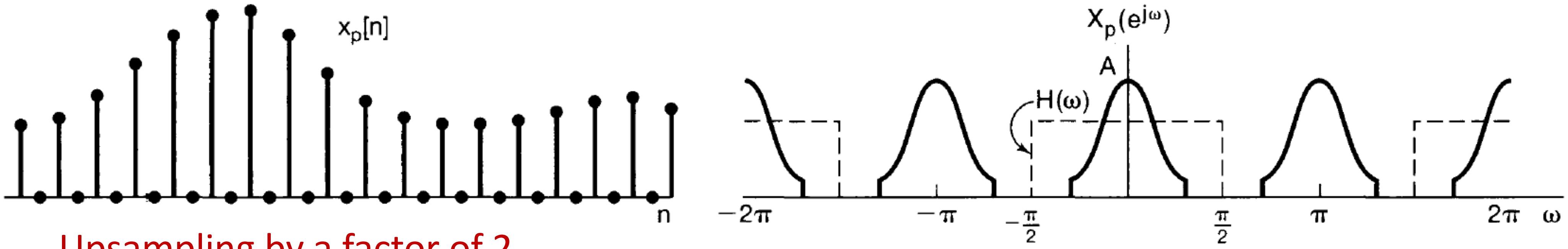


# Sampling of

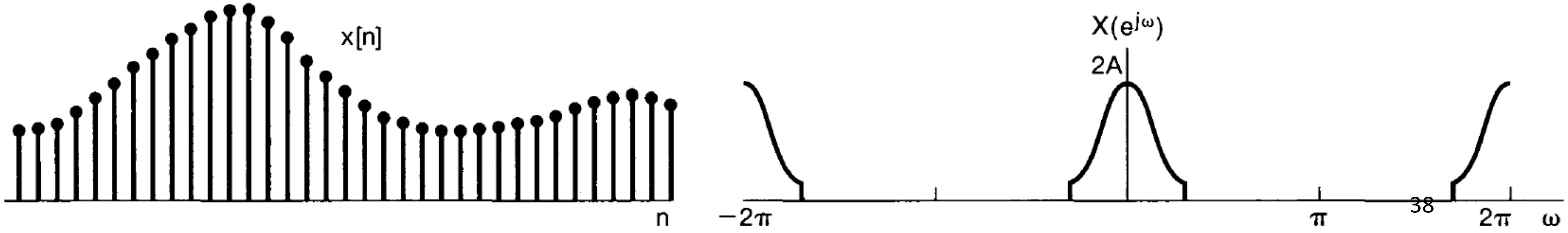
***Interpolation (SRI)***



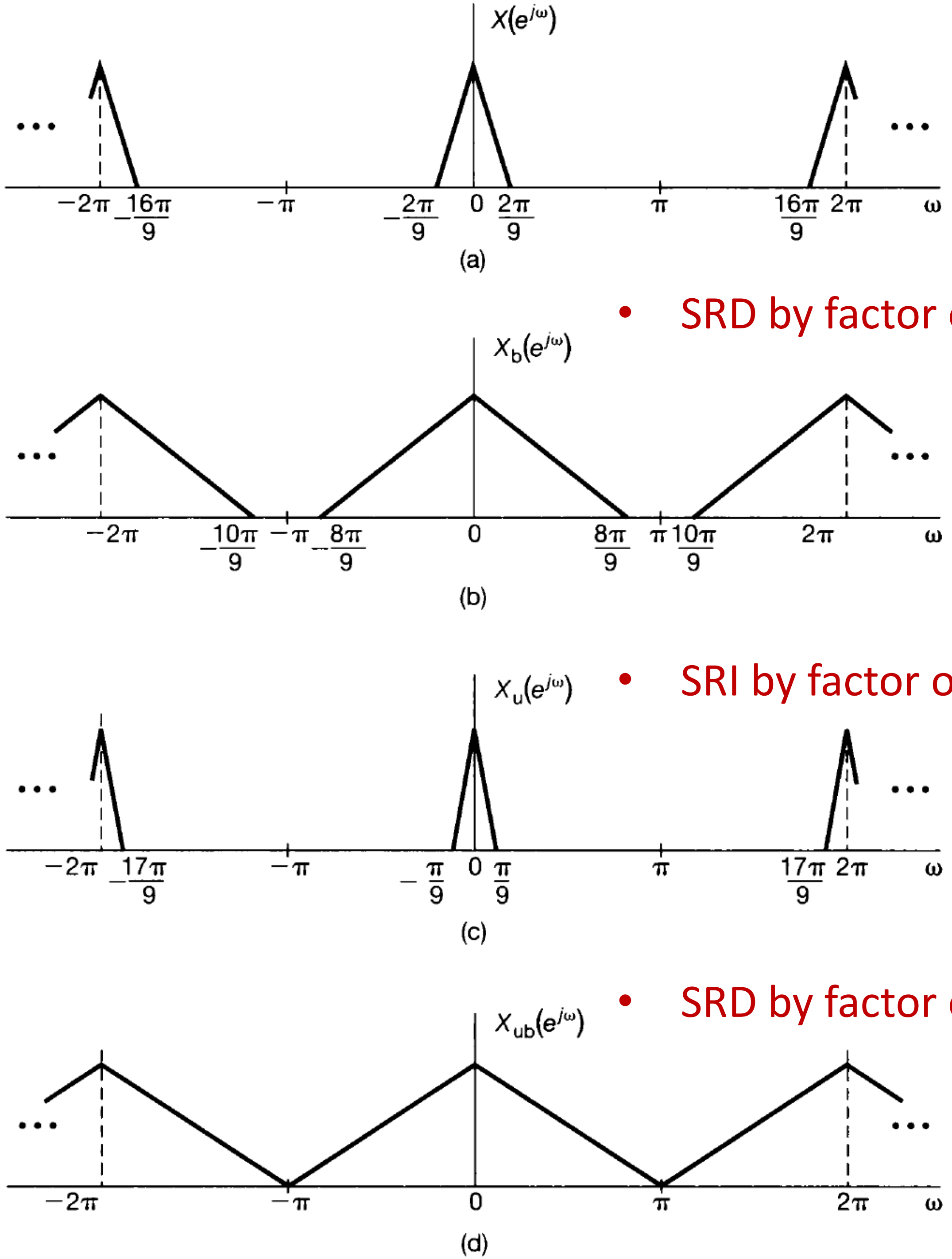

- Prevent mirrors by LPF after SRI $\Longrightarrow$ Interpolator




Upsampling by a factor of 2



## *Interpolation (SRI)*




- SRD by factor of 4
- SRI by factor of 8
- SRD by factor of 9

- Overall, SRD by factor of 4.5