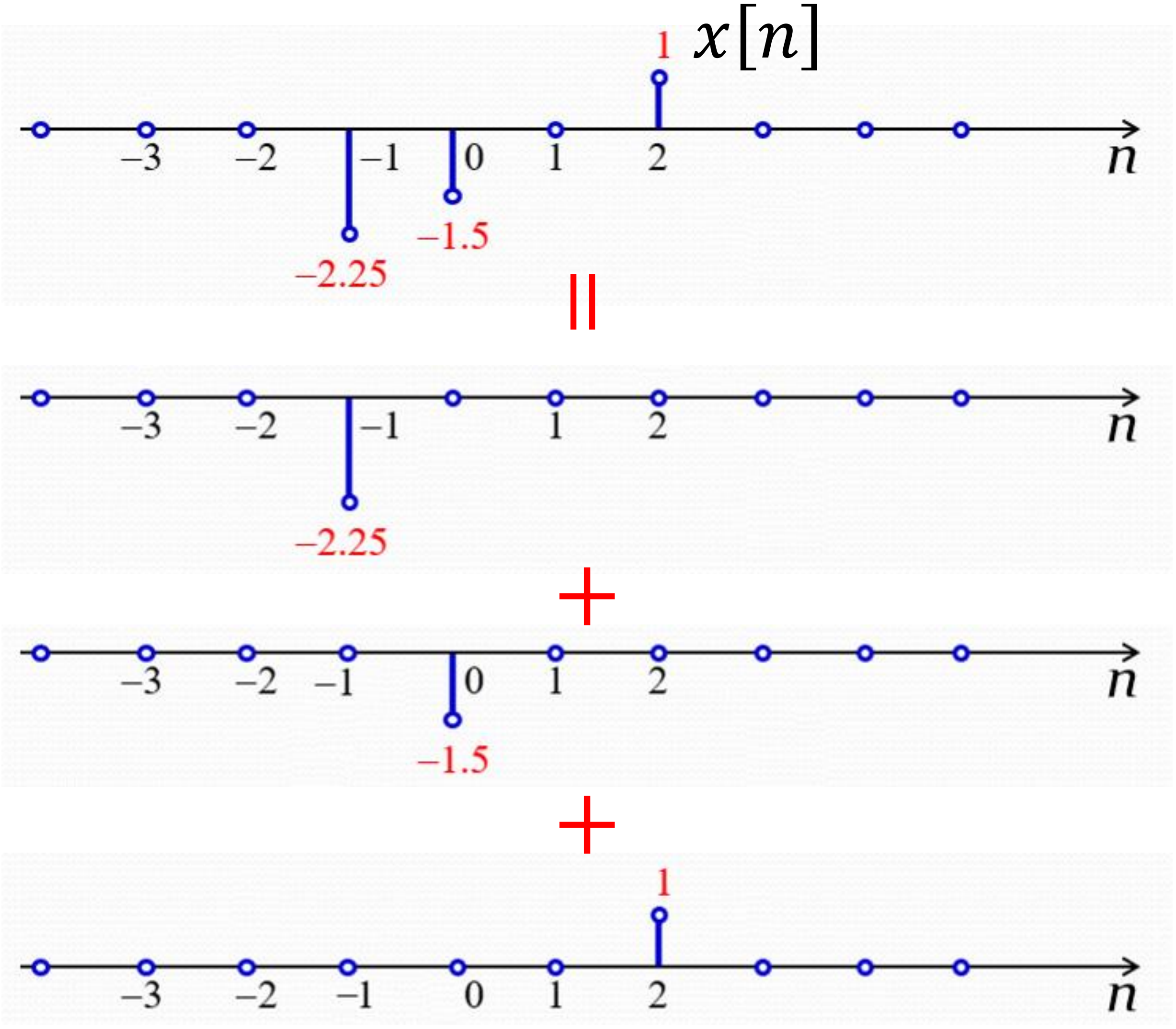
Linear Time-Invariant Systems (ch.2)

- ❑ Discrete-Time LTI Systems
- ❑ Continuous-Time LTI Systems
- ❑ Properties of LTI Systems
- ❑ Differential or Difference Equations

Representation of Discrete-Time Signals in Terms of Impulse



$x[n]$

$=$

$\leftrightarrow x_1[n] = -2.25 \times \delta[n+1]$

$+$

$\leftrightarrow x_2[n] = -1.5 \times \delta[n]$

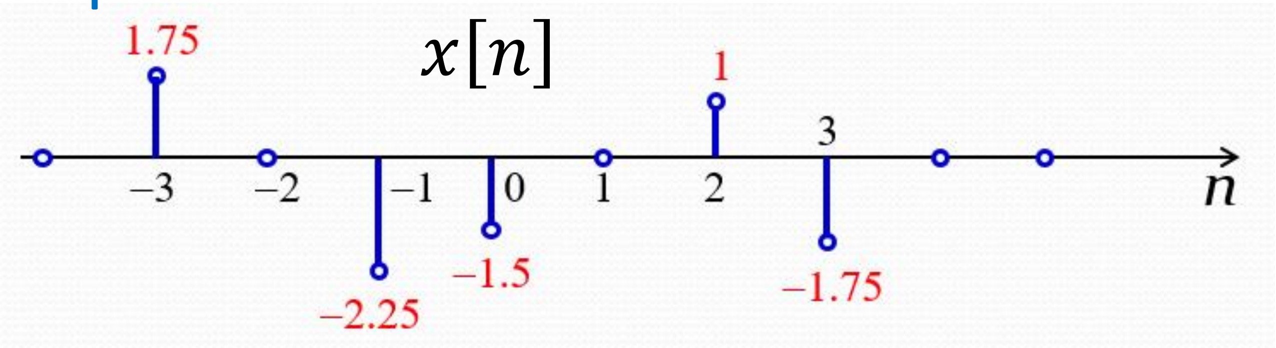
$+$

$\leftrightarrow x_3[n] = 1 \times \delta[n-2]$

Discrete-Time LTI Systems

Representation of Discrete-Time Signals in Terms of Impulse

- An arbitrary sequence can be represented as the weighted sum of shifted unit impulses



$$x[n]=1.75\delta[n+3]-2.25\delta[n+1]-1.5\delta[n]+\delta[n-2]-1.75\delta[n-3]$$

- A general form

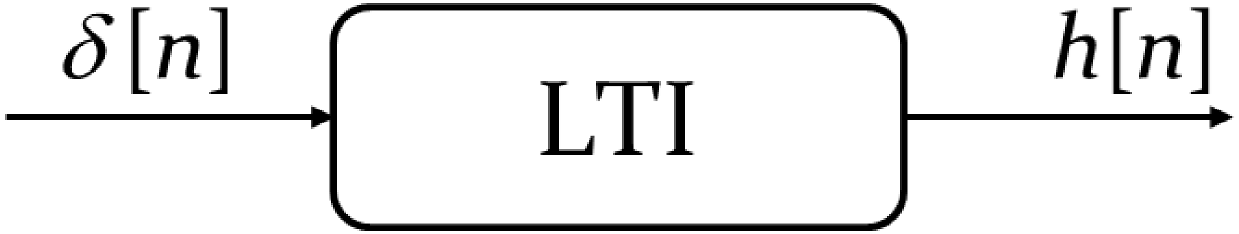
$$x[n]=\sum_{k=-\infty}^{\infty}x[k]\delta[n-k]$$

Sifting property of $\delta[n]$

Discrete-Time LTI Systems

Discrete-Time Unit Impulse Response and the Convolution-Sum

- The response of a system to a unit impulse sequence $\delta[n]$ is called impulse response, denoted by $h[n]$



Discrete-Time Unit Impulse Response and the Convolution-Sum

❑ How to calculate the impulse response of a system

- For any system whose input-output relationship is defined by

$$y[n] = f\{x[n]\}$$

the impulse response $h[n]$ is calculated as

$$h[n] = f\{\delta[n]\}$$ replace $x[n]$ by $\delta[n]$

Discrete-Time Unit Impulse Response and the Convolution-Sum

❑ How to calculate the impulse response of a system

- Examples: a system is defined as

$$y[n] = a_1 x[n] + a_2 x[n-1] + a_3 x[n-2] + a_4 x[n-3]$$

its impulse response $h[n]$ is

$$h[n] = a_1 \delta[n] + a_2 \delta[n-1] + a_3 \delta[n-2] + a_4 \delta[n-3]$$

Discrete-Time Unit Impulse Response and the Convolution-Sum

- How to calculate the impulse response of a system

- Examples: a system is defined as

$$y[n] = \sum_{k=-\infty}^{n} x[k]$$

its impulse response $h[n]$ is

$$h[n] = \sum_{k=-\infty}^{n} \delta[k]$$

Discrete-Time Unit Impulse Response and the Convolution-Sum

❑ How to calculate the impulse response of a system

- Examples: a system is defined as

$$y[n] = x_u[n-1] + \frac{1}{2}(x_u[n-2] + x_u[n])$$

its impulse response $h[n]$ is

$$h[n] = \delta[n-1] + \frac{1}{2}(\delta[n-2] + \delta[n])$$

Discrete-Time LTI Systems

Discrete-Time Unit Impulse Response and the Convolution-Sum

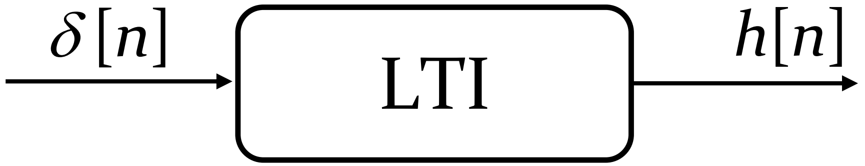
- ❑ An LTI discrete system is completely characterized by its impulse response

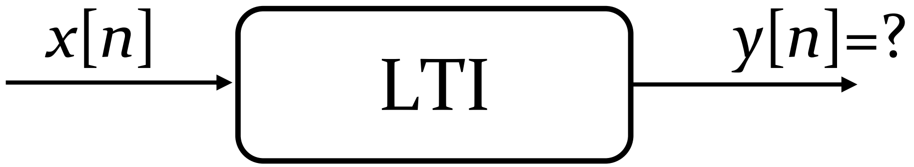
- ❑ In other words, knowing the impulse response one can compute the output of the LTI system for an arbitrary input

Discrete-Time LTI Systems

Discrete-Time Unit Impulse Response and the Convolution-Sum

- The impulse response completely characterizes an LTI system





- Recall, an arbitrary input $x[n]$ can be expressed as a linear combination of shifted unit impulses

$$x[n] = \sum_{k=-\infty}^{\infty} x[k]\delta[n-k]$$

Discrete-Time LTI Systems

Discrete-Time Unit Impulse Response and the Convolution-Sum

- For any $k = k_0$

$$\delta[n] \longrightarrow \boxed{\text{LTI}} \longrightarrow h[n]$$

$$\delta[n-k_0] \longrightarrow \boxed{\text{LTI}} \longrightarrow h[n-k_0]$$

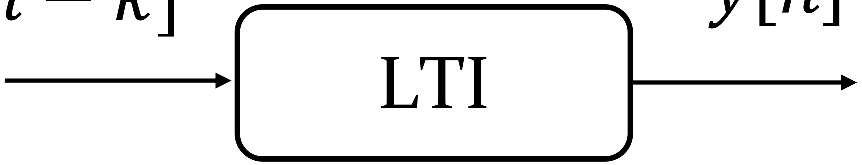
$$x[k_0]\delta[n-k_0] \longrightarrow \boxed{\text{LTI}} \longrightarrow x[k_0]h[n-k_0]$$

$$\Rightarrow \quad x[n] = \sum_{k=-\infty}^{\infty} x[k]\delta[n-k] \longrightarrow \boxed{\text{LTI}} \longrightarrow y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

Discrete-Time LTI Systems

Discrete-Time Unit Impulse Response and the Convolution-Sum

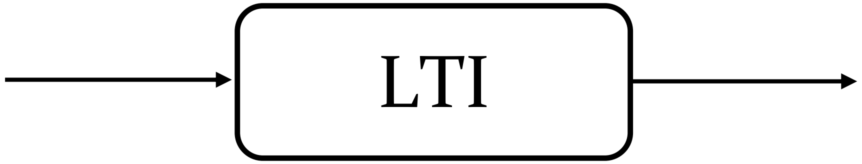
$$x[n] = \sum_{k=-\infty}^{\infty} x[k]\delta[n-k]$$



$$y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

- $\sum_{k=-\infty}^{\infty} x[k]h[n-k]$ is refereed to as the convolution-sum

$x[n]$

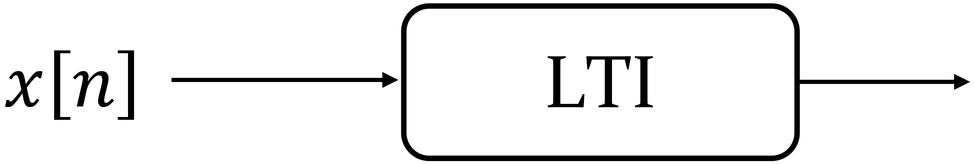


$y[n] = x[n] * h[n]$

Discrete-Time LTI Systems

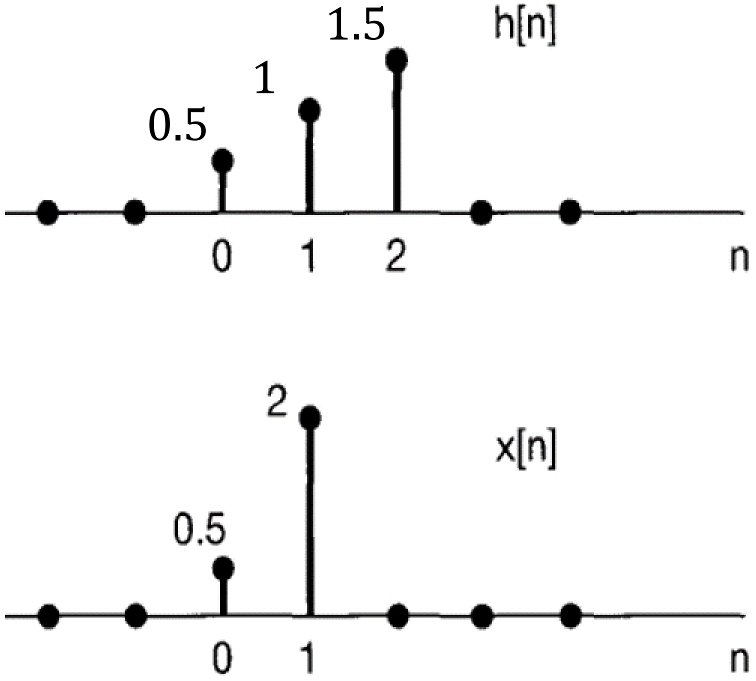
Discrete-Time Unit Impulse Response and the Convolution-Sum

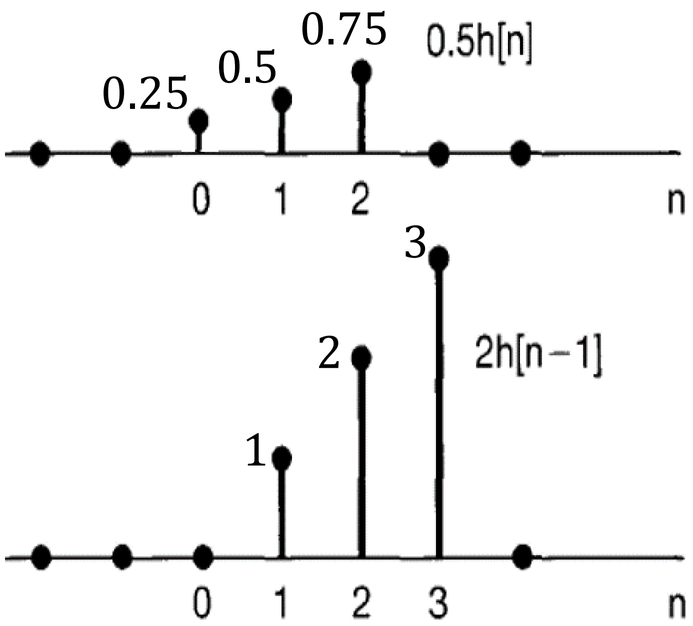
- Convolution-Sum calculation – Method 1: sum of k shifted and scaled h[n]

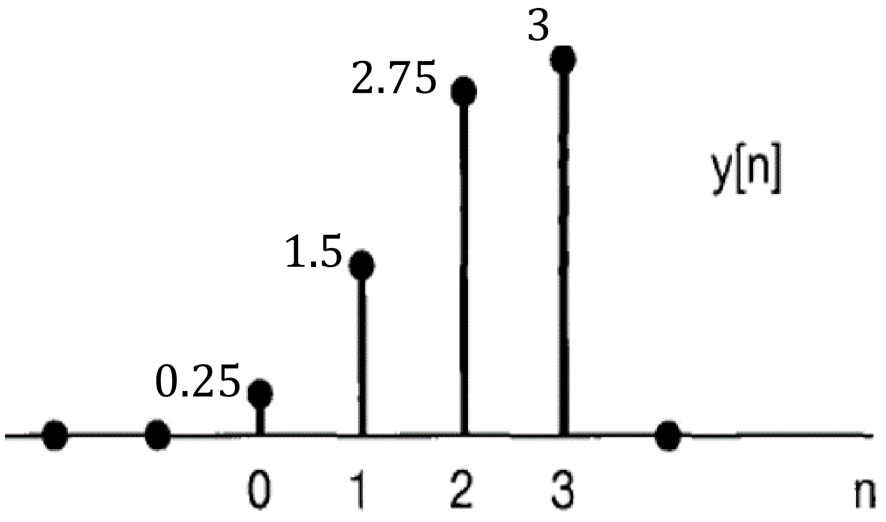


$$y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k] = x[n] * h[n]$$

n: variable, k:constant







Discrete-Time Unit Impulse Response and the Convolution-Sum

- Convolution-Sum calculation–Method 2: calculate y[n] for each n

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k] = x[n] * h[n]$$

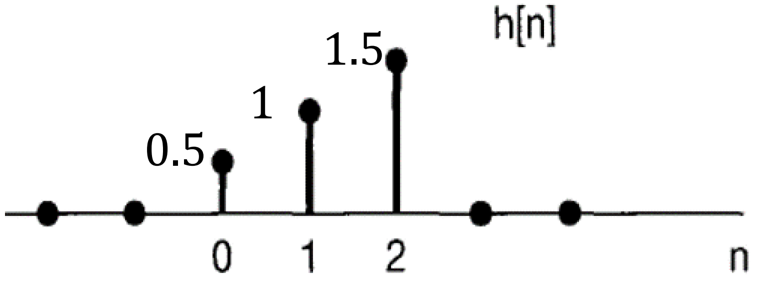
- Step 1: determine the range of k

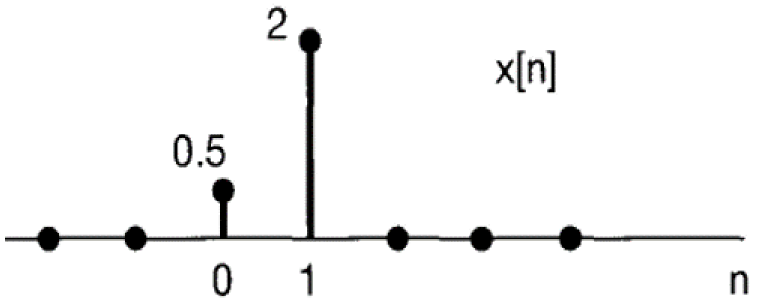
$$k \in \{0,1\}$$

- Step 2: determine the range of n

$$[n-k] \in \{0,1,2\} \leftrightarrow n \in \{0,1,2,3\},$$

For other n, $y[n]=0$





Discrete-Time Unit Impulse Response and the Convolution-Sum

- Convolution-Sum calculation–Method 2: calculate y[n] for each n

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k] = x[n] * h[n]$$

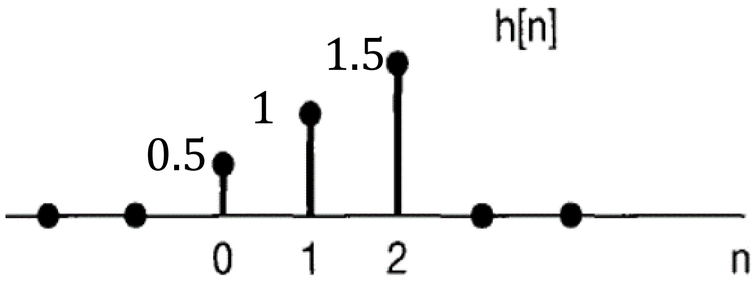
- Step 3: calculate y[n] for each n

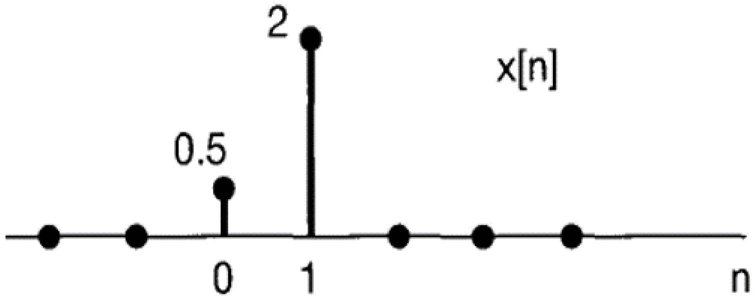
$$y[0] = \sum_{k=0}^{1} x[k]h[0-k] = x[0]h[0] + x[1]h[-1] = 0.25$$

$$y[1] = \sum_{k=0}^{1} x[k]h[1-k] = x[0]h[1] + x[1]h[0] = 1.5$$

$$y[2] = \sum_{k=0}^{1} x[k]h[2-k] = x[0]h[2] + x[1]h[1] = 2.75$$

$$y[3] = \sum_{k=0}^{1} x[k]h[3-k] = x[0]h[3] + x[1]h[2] = 3$$





Discrete-Time Unit Impulse Response and the Convolution-Sum

Convolution-Sum calculation–Method 3

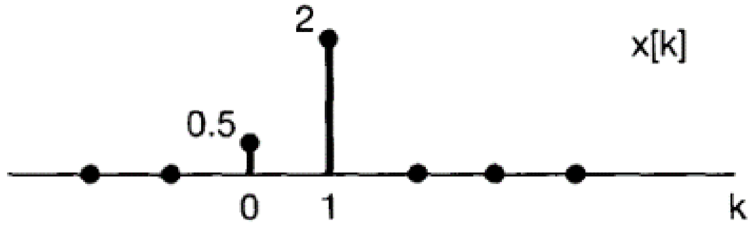
$$y[n] = x[n] * h[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

For each n :

- Step 1: change time variables $x[n] \rightarrow x[k]$, $h[n] \rightarrow h[k]$, and reverse $h[k] \rightarrow h[-k]$
- Step 2: Shift $h[-k] \rightarrow h[n-k]$, n is considered as a constant
- Step 3: multiply $x[k] \cdot h[n-k]$
- Step 4: Summation $\sum_{k=-\infty}^{\infty} x[k] \cdot h[n-k]$

Change n, repeat step 1 to 4, calculate another $y[n]$

Discrete-Time LTI Systems

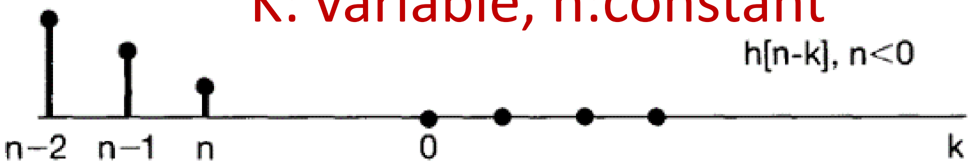


The Convolution-Sum

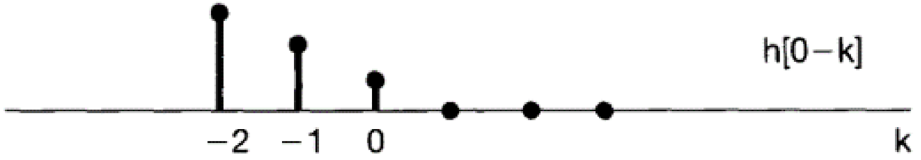
K: variable, n:constant

- Convolution-Sum calculation – Method 3
  - If the lengths of the two sequences are *M* and *N*, then the sequence generated by the convolution is of length *M*+*N*–1

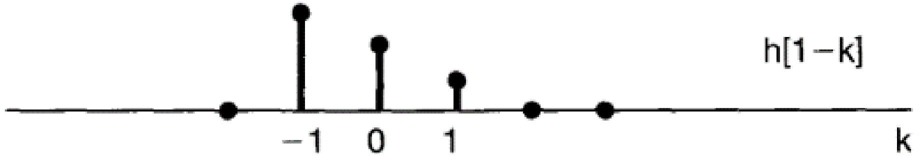
$y[n] = 0$, for $n<0$



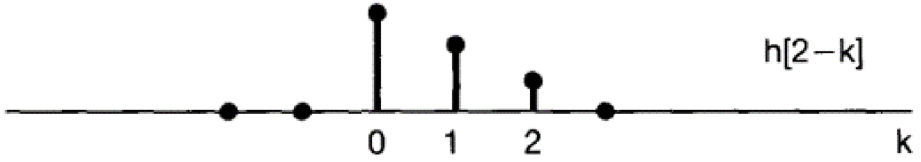
$$y[0] = \sum_{k=0}^{1} x[k]h[0-k]$$



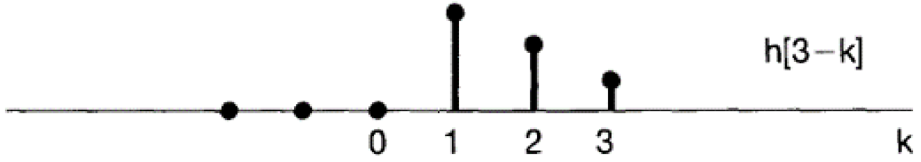
$$y[1] = \sum_{k=0}^{1} x[k]h[1-k]$$



$$y[2] = \sum_{k=0}^{1} x[k]h[2-k]$$



$$y[3] = \sum_{k=0}^{1} x[k]h[3-k]$$



$y[n] = 0$, for $n>3$

Discrete-Time LTI Systems

The Convolution-Sum

- Examples

$$y[n] = x[n] * \delta[n] = \sum_{k=-\infty}^{\infty} x[k]\delta[n-k] = x[n]$$

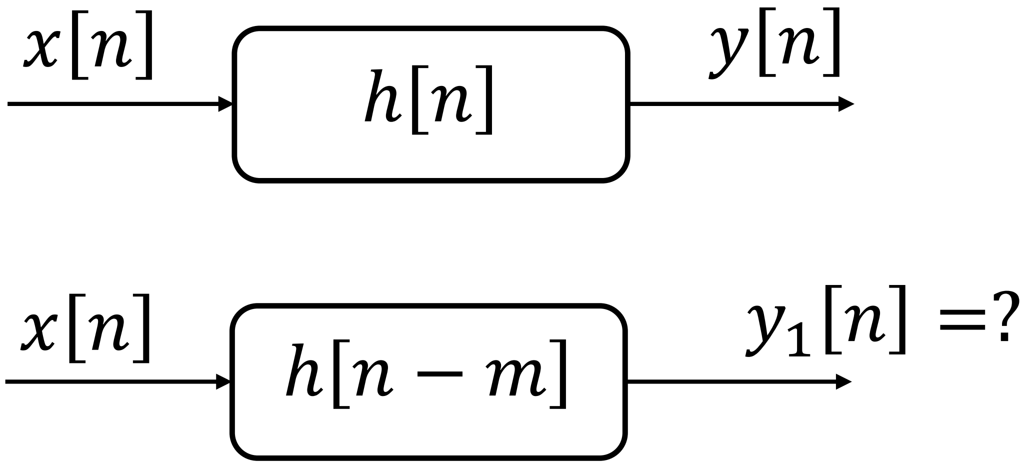
$$y[n] = x[n] * \delta[n-d] = \sum_{k=-\infty}^{\infty} x[k]\delta[n-k-d]$$ Let $k + d = k'$

$$= \sum_{k'=-\infty}^{\infty} x[k'-d]\delta[n-k']$$

$$= x[n-d] * \delta[n] = x[n-d]$$

The Convolution-Sum

- Examples



$$y_1[n] = x[n] * h[n-m] = \sum_{k=-\infty}^{\infty} x[k]h[n-k-m]$$

Let $k + m = k'$

$$= \sum_{k'=-\infty}^{\infty} x[k'-m]h[n-k']$$

$$= x[n-m] * h[n] = y[n-m]$$

Linear Time-Invariant Systems (ch.2)

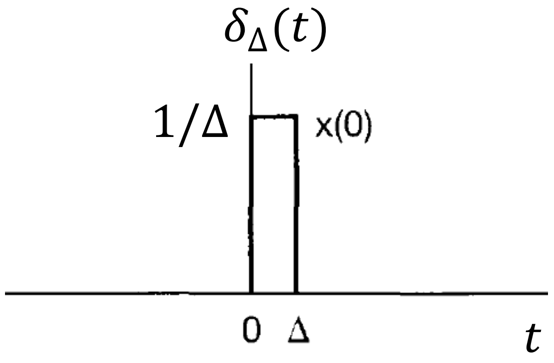
- ❑ Discrete-Time LTI Systems
- ❑ Continuous-Time LTI Systems
- ❑ Properties of LTI Systems
- ❑ Differential or Difference Equations

Continuous-Time LTI Systems

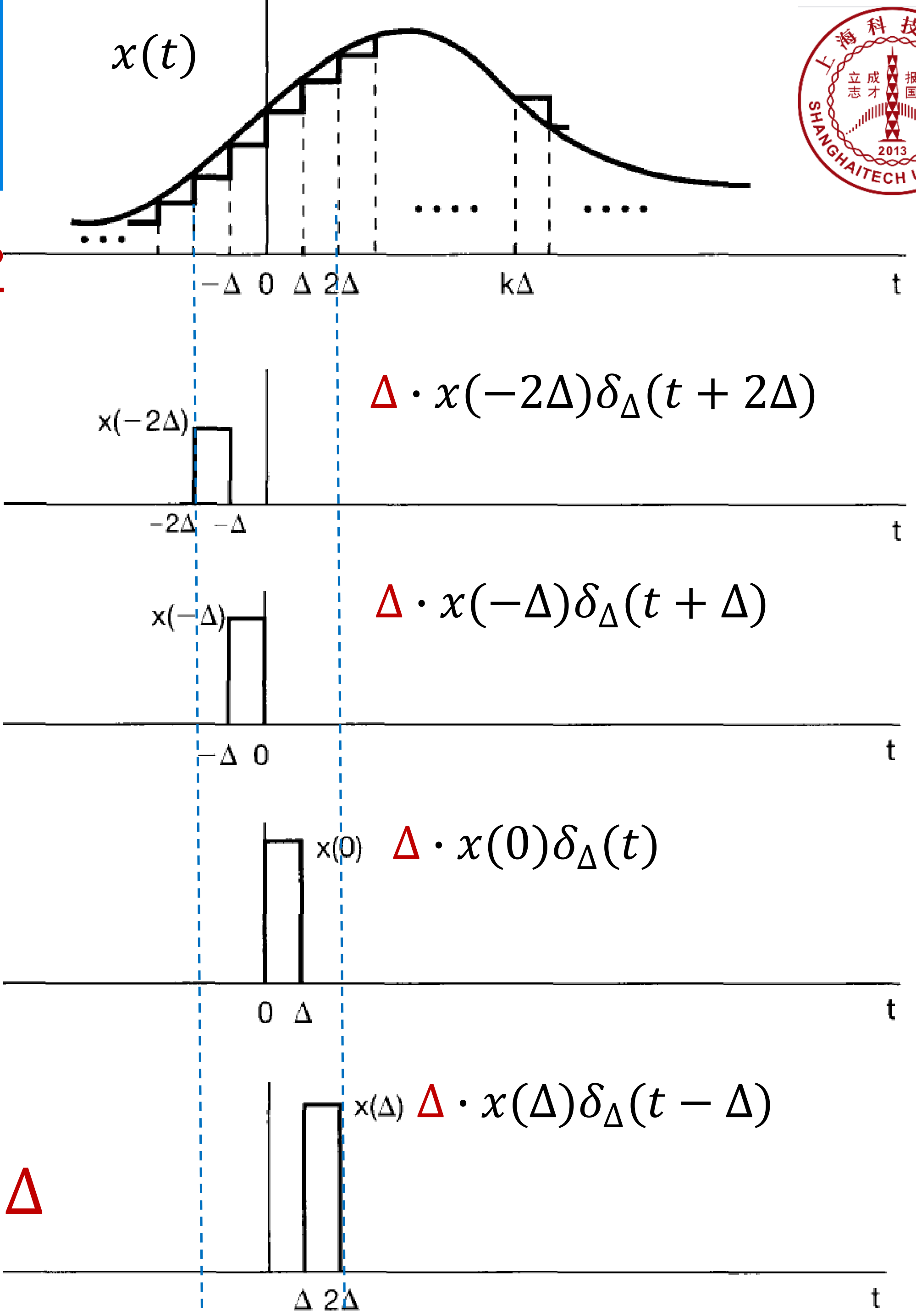
Continuous-Time Signals in Terms of Impulse

- "staircase" approximation of $x(t)$

$$\delta_\Delta(t) = \begin{cases} \frac{1}{\Delta}, & 0 \le t \le \Delta \\ 0, & otherwise \end{cases}$$



$$\hat{x}(t) = \sum_{k=-\infty}^{\infty} x(k\Delta)\delta_\Delta(t-k\Delta)\cdot\Delta$$



Continuous-Time LTI Systems

Continuous-Time Signals in Terms of Impulse

- "staircase" approximation of $x(t)$

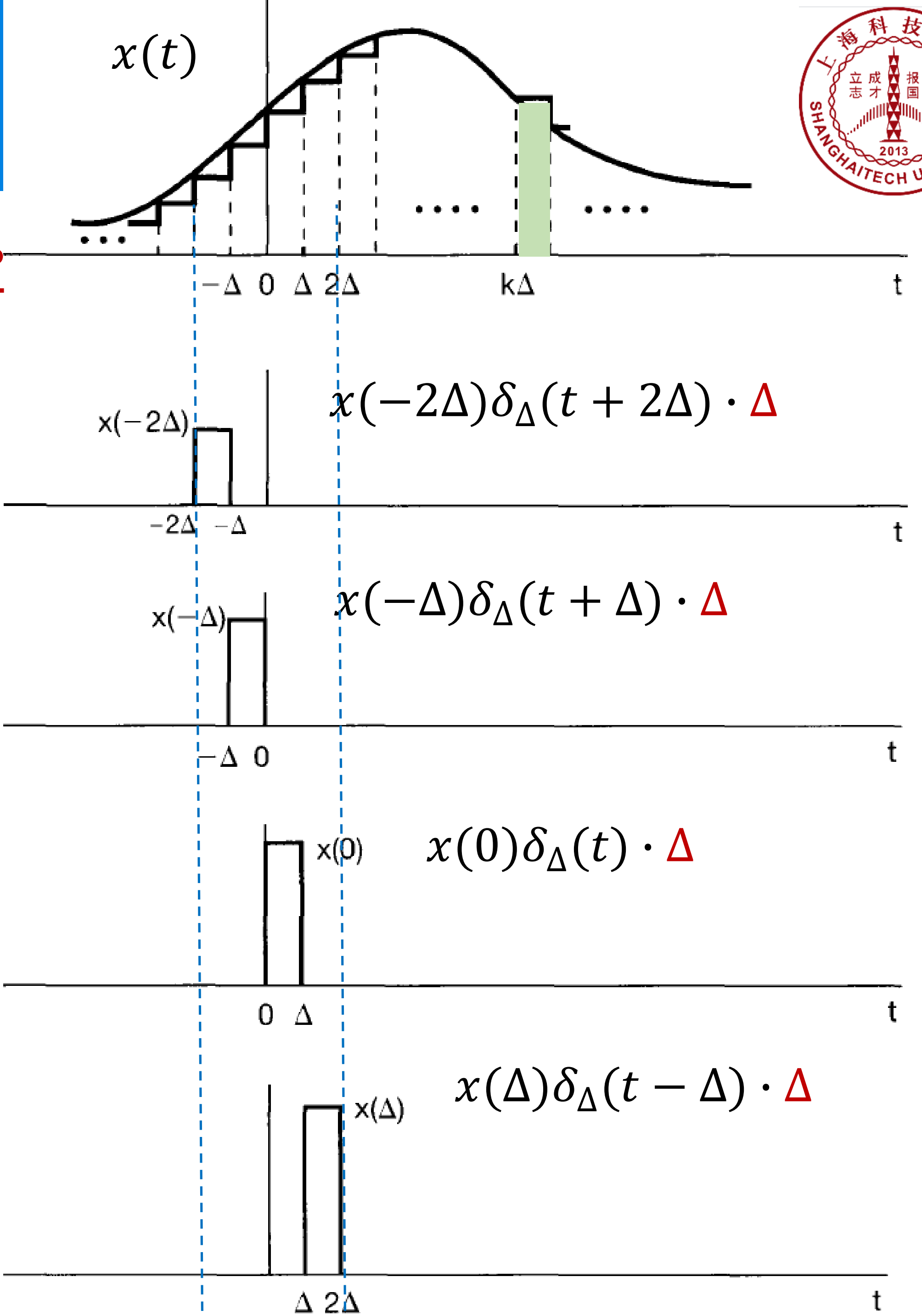
$$\hat{x}(t) = \sum_{k=-\infty}^{\infty} x(k\Delta)\delta_\Delta(t-k\Delta)\cdot\Delta$$

$$\Delta \to 0,\ \hat{x}(t) \to x(t),$$

$$x(k\Delta) \to x(\tau), \delta_\Delta(t-k\Delta) \to \delta(t-\tau)$$

$$x(t) = \lim_{\Delta\to 0} \hat{x}(t) = \int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)\, d\tau$$

Sifting property of $\delta(t)$



Continuous-Time LTI Systems

Continuous-Time Signals in Terms of Impulse

- Using sampling property of $\delta(t)$

$$\int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)\, d\tau = ?$$

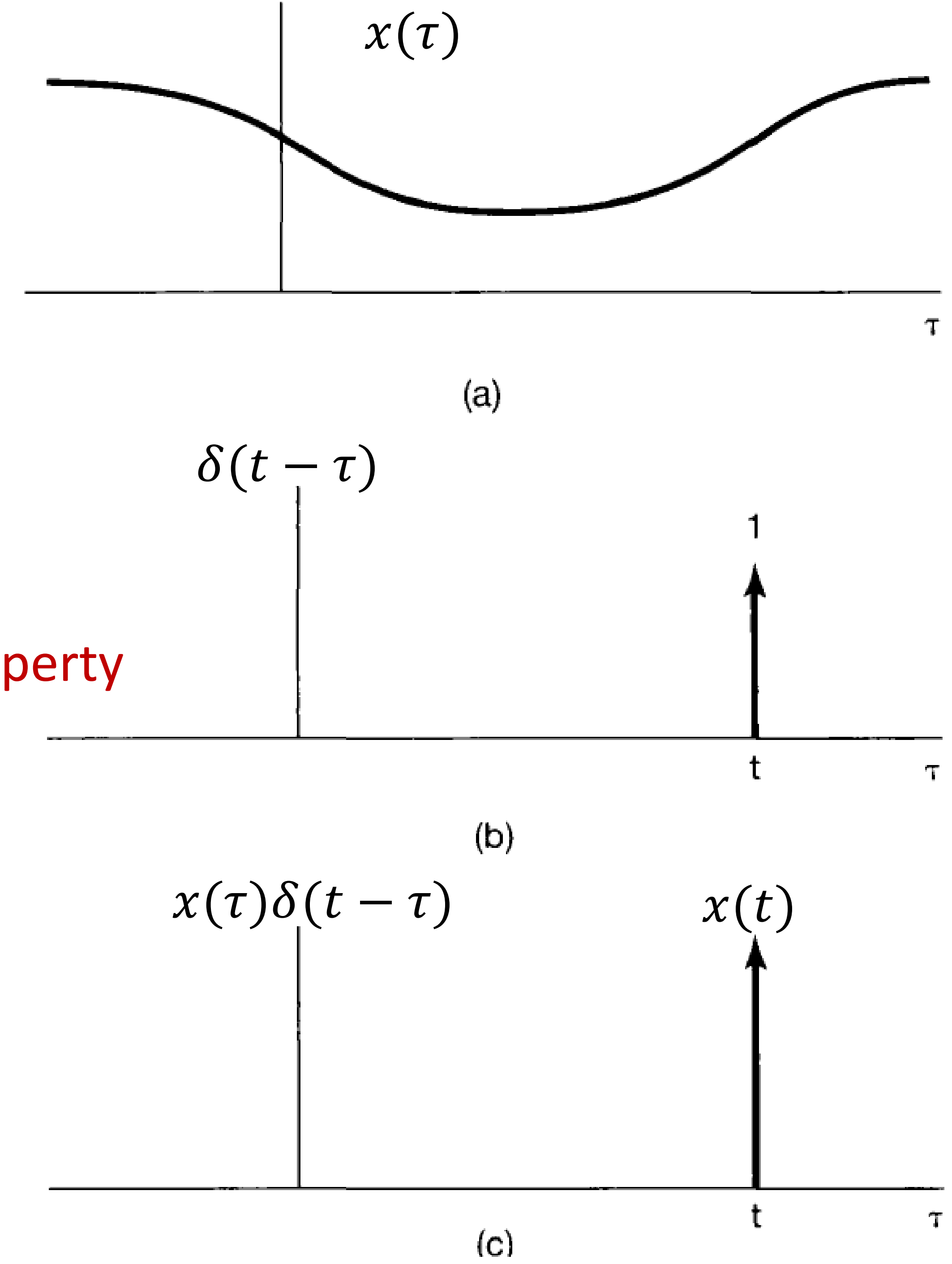
$$x(t)\delta(t-t_0) = x(t_0)\delta(t-t_0)$$ sampling property

$$x(\tau)\delta(t-\tau) = x(t)\delta(t-\tau)$$ t: constant

$$\therefore \int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)\, d\tau = \int_{-\infty}^{\infty} x(t)\delta(t-\tau)\, d\tau$$

$$= x(t)\int_{-\infty}^{\infty} \delta(t-\tau)\, d\tau$$

$$= x(t)$$



Continuous-Time LTI Systems

Continuous-Time Signals in Terms of Impulse

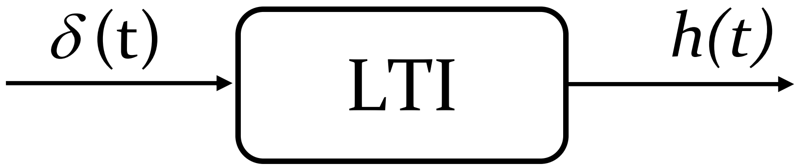
- An example

$$x(t) = \int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)\, d\tau$$

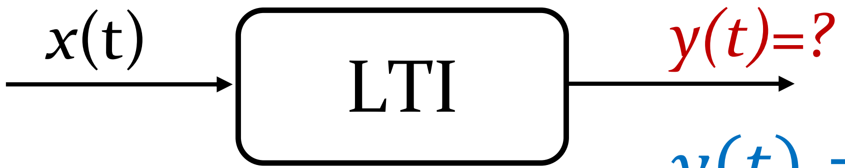
$$u(t) = \int_{-\infty}^{\infty} u(\tau)\delta(t-\tau)\, d\tau = \int_{0}^{\infty} \delta(t-\tau)\, d\tau$$

Continuous-Time Unit Impulse Response and Convolution Integral

- Continuous-Time Unit Impulse Response



- What about



$$x(t) = \int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)\, d\tau$$

Sum of weighted and shifted impulses

$$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)\, d\tau$$

Sum of weighted and shifted impulse response

- Convolution integral

$$\int_{-\infty}^{\infty} x(\tau)h(t-\tau)\, d\tau = x(t) * h(t)$$

Continuous-Time Unit Impulse Response and Convolution Integral

- Computation convolution integral

$$x(t) * h(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)\,d\tau$$

- Change time variables $x(t) \rightarrow x(\tau)$, $h(t) \rightarrow h(\tau)$, and reverse $h(\tau) \rightarrow h(-\tau)$
- Shift $h(-\tau) \rightarrow h(t-\tau)$
- Multiply $x(\tau) \cdot h(t-\tau)$
- Integral $\int_{-\infty}^{\infty} x(\tau) \cdot h(t-\tau)\mathrm{d}\tau$

Continuous-Time Unit Impulse Response and Convolution Integral

- Computation convolution integral: examples

$$x(t) * h(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)\, d\tau$$

$$x(t) * \delta(t) = \int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)\, d\tau = x(t)$$

$$x(t) * \delta(t-t_0) = \int_{-\infty}^{\infty} x(\tau)\delta(t-\tau-t_0)\, d\tau = \int_{-\infty}^{\infty} x(\tau)\delta(t-(\tau+t_0))\, d\tau$$

$$= \int_{-\infty}^{\infty} x(\tau'-t_0)\delta(t-\tau')\, d\tau' = x(t-t_0) * \delta(t)$$

$$= x(t-t_0)$$

Continuous-Time Unit Impulse Response and Convolution Integral

- Computation convolution integral: examples

$$x(t)=e^{-at}u(t),\ h(t)=u(t),\ \ a>0 \quad x(t)*h(t)=?$$

$$y(t)=\int_{-\infty}^{\infty} e^{-a\tau}u(\tau)\cdot u(t-\tau)d\tau$$

For $t<0$ $\quad x(\tau)\cdot h(t-\tau)=0 \quad \Rightarrow \quad y(t)=0$

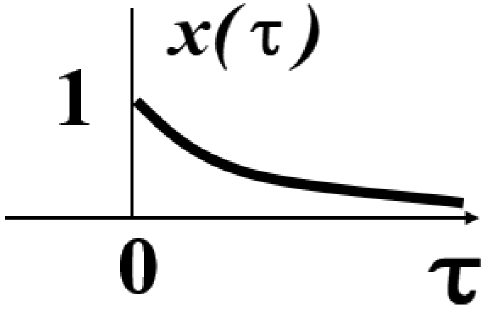
For $t\geq 0$ $\quad y(t)=\int_0^t e^{-a\tau}d\tau=\frac{-1}{a}e^{-at}\Big|_0^t=\frac{1}{a}(1-e^{-at})$

Continuous-Time Unit Impulse Response and Convolution Integral

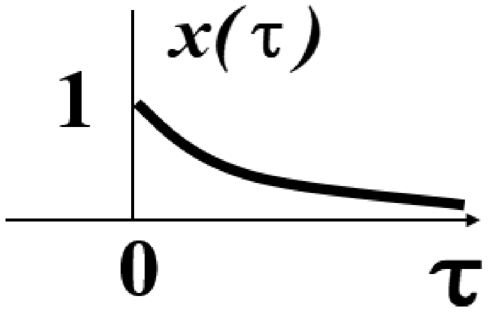
- Computation convolution integral: Graphical Solution

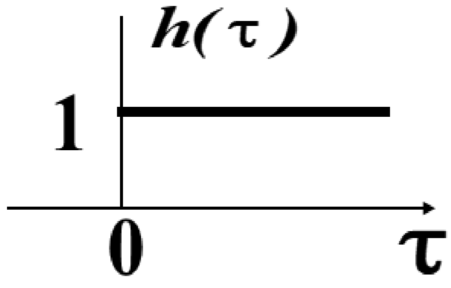
$$x(t) = e^{-at}u(t),\ h(t) = u(t),\ \ a > 0$$

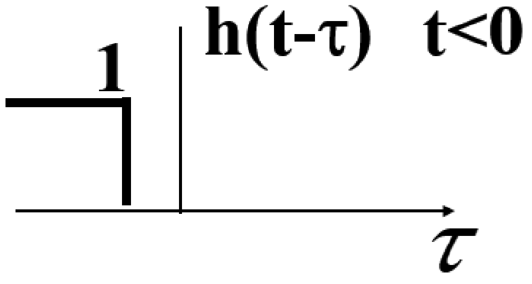
$$x(t) * h(t) = ?$$

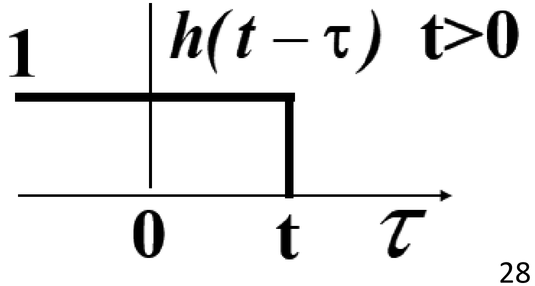


τ : variable, t: constant









$$y(t) = \int_0^t e^{-a\tau} d\tau = \frac{-1}{a} e^{-at} \Big|_0^t = \frac{1}{a}(1 - e^{-at})$$

Continuous-Time Unit Impulse Response and Convolution Integral

- Computation convolution integral: examples

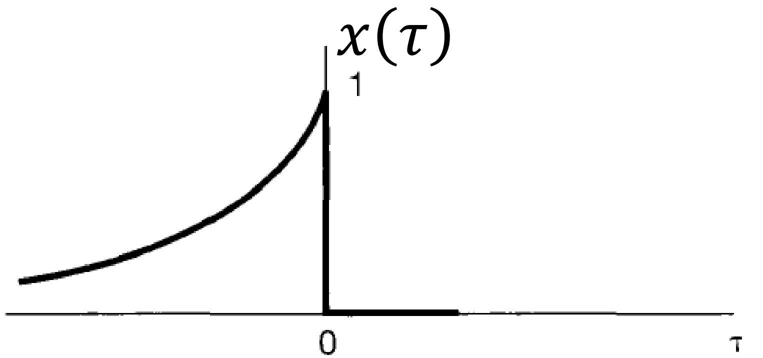
$$x(t) = e^{2t}u(-t) \quad h(t) = u(t-3) \quad x(t) * h(t) = ?$$

For $t - 3 \leq 0$

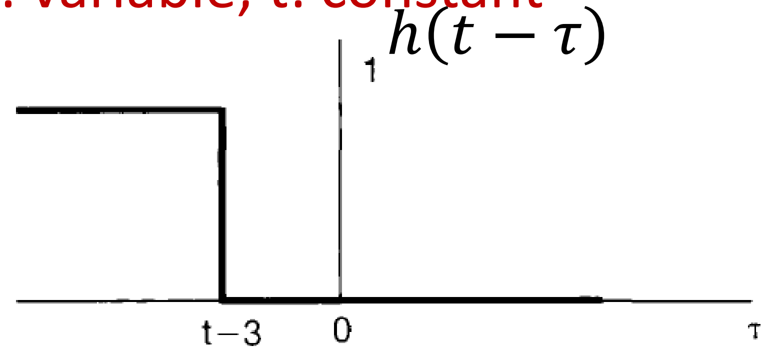
$$x(t) * h(t) = \int_{-\infty}^{t-3} e^{2\tau} d\tau = \frac{1}{2} e^{2(t-3)}$$

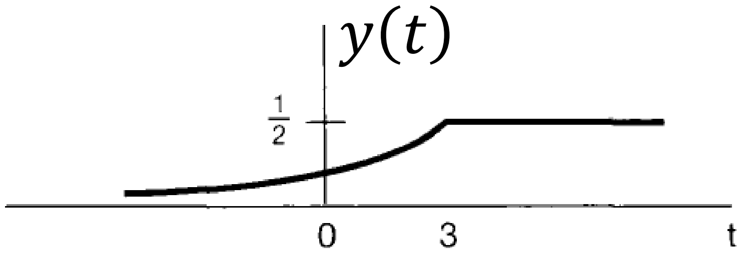
For $t - 3 \geq 0$

$$x(t) * h(t) = \int_{-\infty}^{0} e^{2\tau} d\tau = \frac{1}{2}$$



τ : variable, t: constant



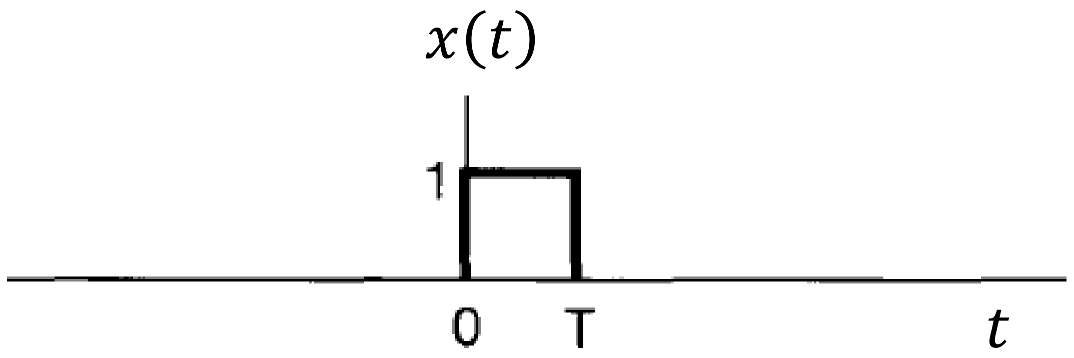


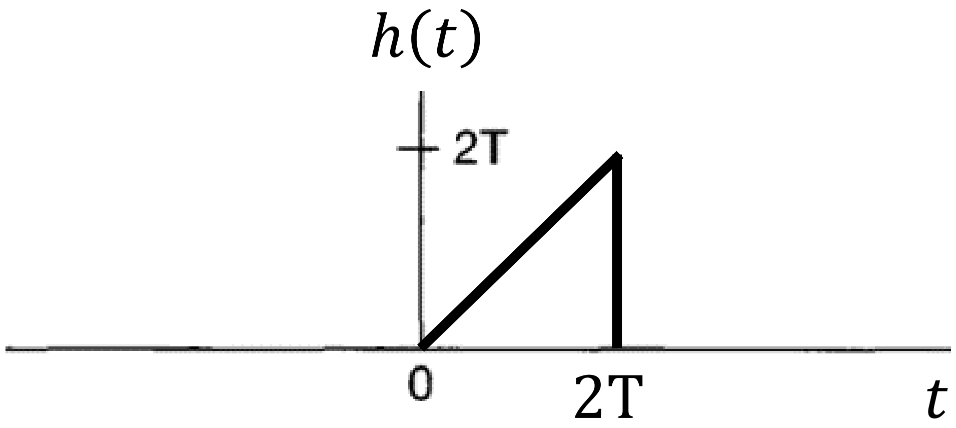
Continuous-Time LTI Systems

Continuous-Time Unit Impulse Response and Convolution Integral

- Computation convolution integral: examples

$$x(t) = \begin{cases} 1, 0 < t < T \\ 0, \text{otherwise} \end{cases} \qquad h(t) = \begin{cases} t, 0 < t < 2T \\ 0, \text{otherwise} \end{cases}$$





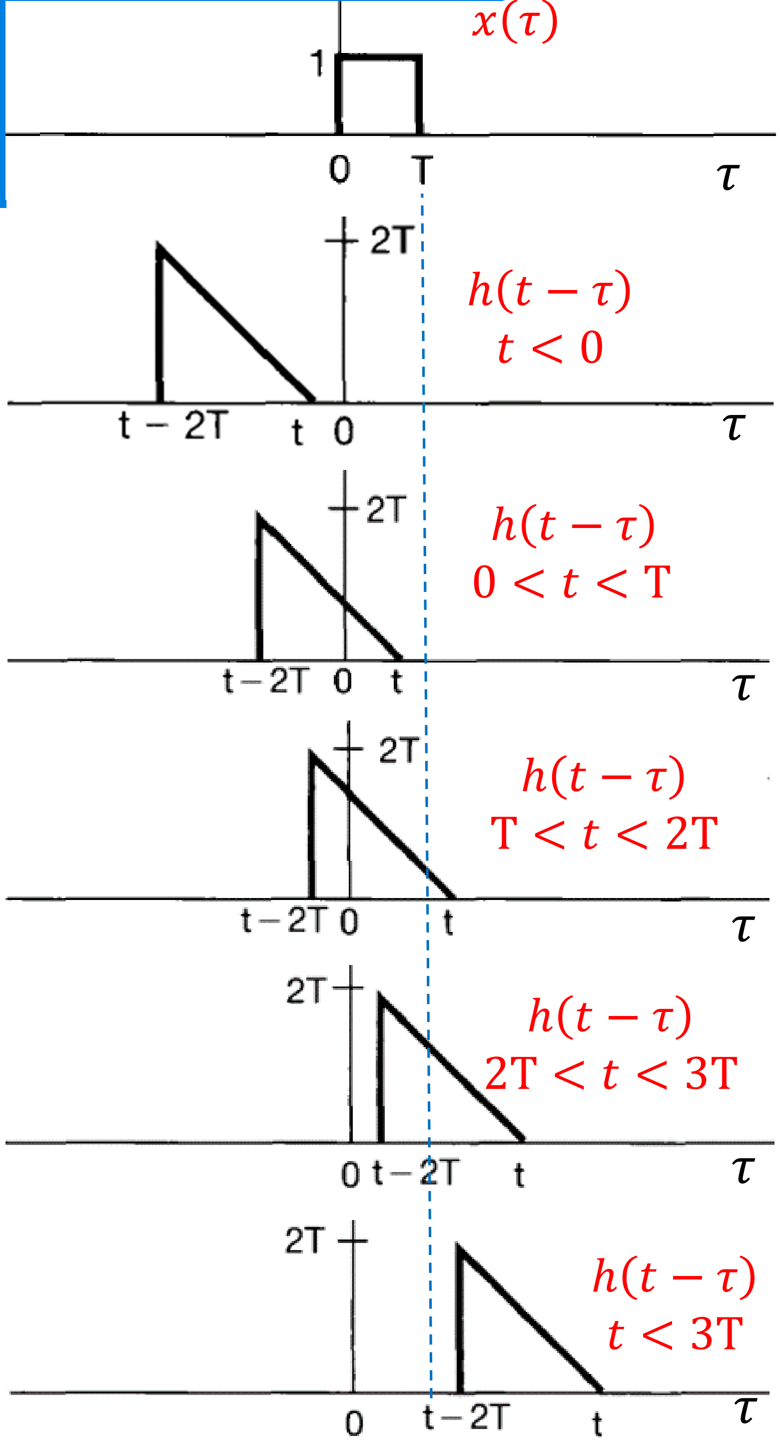
$$x(t) * h(t) = ?$$

Continuous-Time LTI Systems

Convolution Integral

- Computation: examples

$$y(t) = \begin{cases} 0, t < 0 \\ \int_0^t (t-\tau)d\tau = \frac{1}{2}t^2, 0 < t < T \\ \int_0^T (t-\tau)d\tau = Tt - \frac{1}{2}T^2, T < t < 2T \\ \int_{t-2T}^T (t-\tau)d\tau = -\frac{1}{2}t^2 + Tt + \frac{3}{2}T^2, 2T < t < 3T \\ 0, t > 3T \end{cases}$$



Linear Time-Invariant Systems (ch.2)

- ❑ Discrete-Time LTI Systems
- ❑ Continuous-Time LTI Systems
- ❑ Properties of LTI Systems
- ❑ Differential or Difference Equations

Properties of LTI Systems

The commutative property

- Discrete-time $x[n] * h[n] = h[n] * x[n]$

$$x[n] * h[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k] \overset{n-k=m}{=} \sum_{m=-\infty}^{\infty} h[m]x[n-m] = h[n] * x[n]$$

- Continuous-time $x(t) * h(t) = h(t) * x(t)$

$$x(t) * h(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)\,d\tau \overset{t-\tau=\tau'}{=} \int_{-\infty}^{\infty} h(\tau')x(t-\tau')d\tau' = h(t) * x(t)$$

Properties of LTI Systems

The distribute property

- Discrete-time

$$x[n] * (h_1[n] + h_2[n]) = x[n] * h_1[n] + x[n] * h_2[n]$$

- Proof

$$x[n] * (h_1[n] + h_2[n]) = \sum_{k=-\infty}^{\infty} x[k]\,(h_1[n-k] + h_2[n-k])$$

$$= \sum_{k=-\infty}^{\infty} x[k]\,h_1[n-k] + \sum_{k=-\infty}^{\infty} x[k]\,h_2[n-k])$$

$$= x[n] * h_1[n] + x[n] * h_2[n]$$

Properties of LTI Systems

The distribute property

- Continuous-time

$$x(t) * \big(h_1(t) + h_2(t)\big) = x(t) * h_1(t) + x(t) * h_2(t)$$

- Proof

$$x(t) * \big(h_1(t) + h_2(t)\big) = \int_{-\infty}^{\infty} x(\tau)\big(h_1(t-\tau) + h_2(t-\tau)\big)d\tau$$

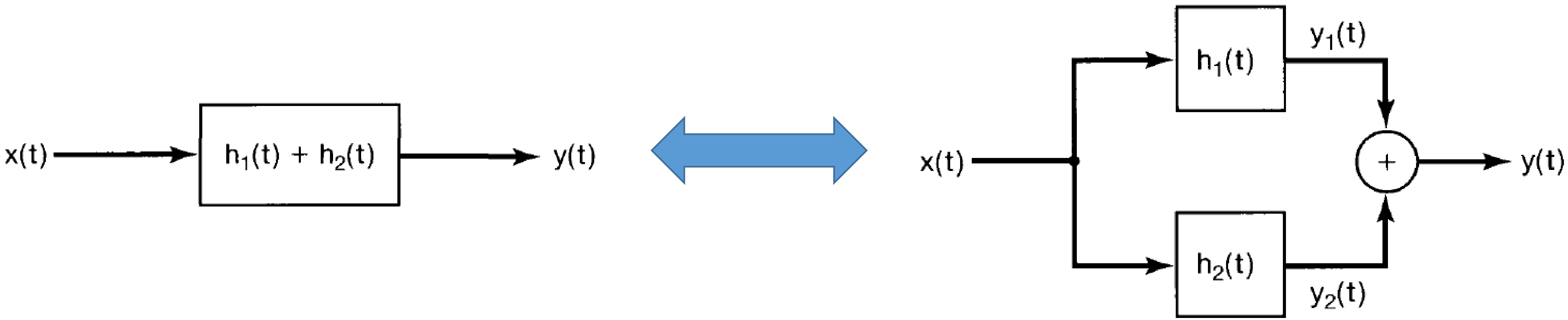
$$= \int_{-\infty}^{\infty} x(\tau)h_1(t-\tau)d\tau + \int_{-\infty}^{\infty} x(\tau)h_2(t-\tau)d\tau$$

$$= x(t) * h_1(t) + x(t) * h_2(t)$$

The distribute property

- Continuous-time

$$x(t) * \left(h_1(t) + h_2(t)\right) = x(t) * h_1(t) + x(t) * h_2(t)$$



Properties of LTI Systems

The associative property

- Discrete-time $x[n] * (h_1[n] * h_2[n]) = (x[n] * h_1[n]) * h_2[n]$

$$x[n] * (h_1[n] * h_2[n]) = x[n] * y[n], \qquad y[n] = \sum_{m=-\infty}^{\infty} h_1[m]h_2[n-m]$$

$$= \sum_{k=-\infty}^{\infty} x[k]y[n-k] = \sum_{k=-\infty}^{\infty} x[k] \sum_{m=-\infty}^{\infty} h_1[m]h_2[n-k-m]$$

Let $k + m = l$

$$= \sum_{k=-\infty}^{\infty} x[k] \sum_{l=-\infty}^{\infty} h_1[l-k]h_2[n-l]$$

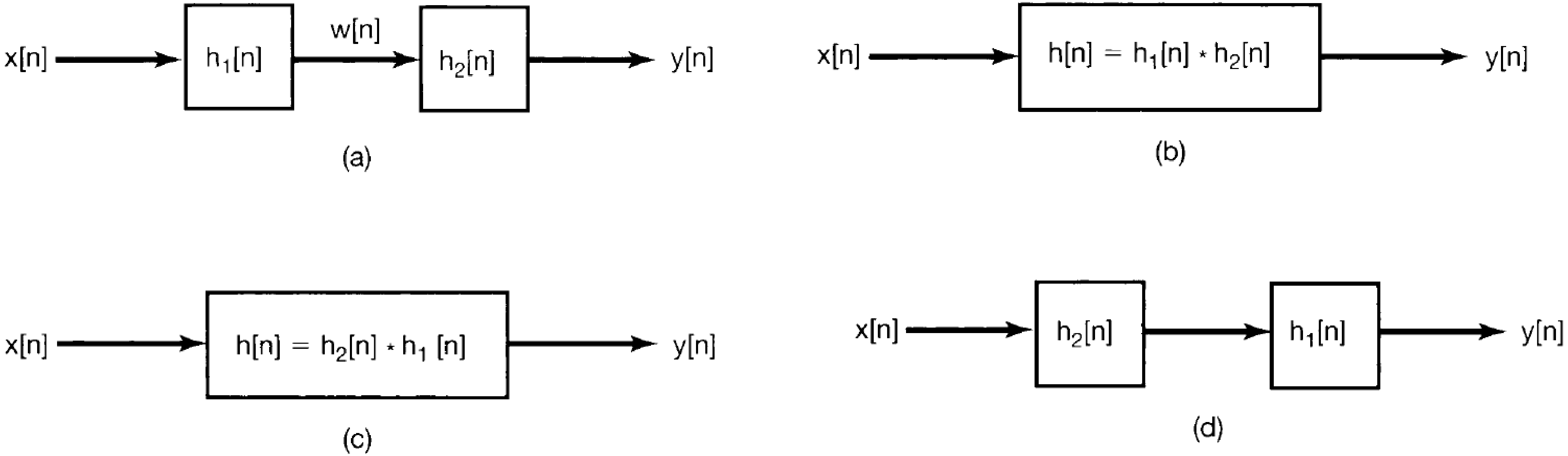
$$= \sum_{l=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} x[k]h_1[l-k]\, h_2[n-l]$$

$$= \sum_{l=-\infty}^{\infty} (x[l] * h_1[l])\, h_2[n-l] = (x[n] * h_1[n]) * h_2[n]$$

Properties of LTI Systems

The associative property

- Discrete-time



Properties of LTI Systems

The associative property

- Continuous-time $$x(t) * \big(h_1(t) * h_2(t)\big) = \big(x(t) * h_1(t)\big) * h_2\,(t)$$

$$x(t) * \big(h_1(t) * h_2(t)\big) = x(t) * \int_{-\infty}^{\infty} h_1(\tau) h_2(t-\tau) d\tau$$

$$= \int_{-\infty}^{\infty} x(\tau') \int_{-\infty}^{\infty} h_1(\tau) h_2(t-\tau'-\tau) d\tau\, d\tau'$$

Let $\tau' + \tau = \tau''$

$$= \int_{-\infty}^{\infty} x(\tau') \int_{-\infty}^{\infty} h_1(\tau''-\tau') h_2(t-\tau'') d\tau''\, d\tau'$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x(\tau') h_1(\tau''-\tau') d\tau'\; h_2(t-\tau'') d\tau''$$

$$= \int_{-\infty}^{\infty} x(\tau'') * h_1(\tau'')\; h_2(t-\tau'') d\tau'' = \big(x(t) * h_1(t)\big) * h_2\,(t)$$

Properties of LTI Systems

LTI systems with and without memory

- Discrete-time system without memory only if $h[n] = 0$ for all $n \neq 0$

$$h[n] = h[0]\delta[n] = k\delta[n] \qquad y[n] = kx[n]$$

Why?

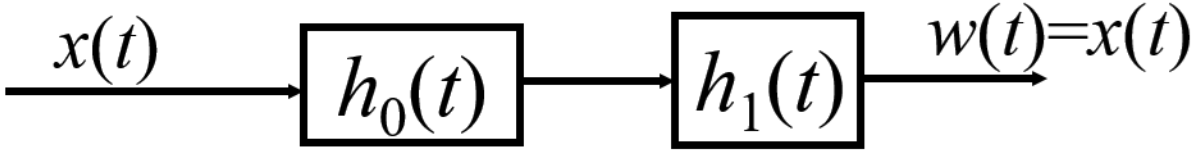
- Continuous-time system without memory only if $h(t) = 0$ for all $t \neq 0$

$$h(t) = h(0)\delta(t) = k\delta(t) \qquad y(t) = kx(t)$$

Properties of LTI Systems

Invertibility for LTI systems

- If $h_0(t) * h_1(t) = \delta(t)$, the system $h_1(t)$ is the inverse of the system $h_0(t)$



- Similarly, if $h_0[n] * h_1[n] = \delta[n]$, the system $h_1[n]$ is the inverse system of $h_0[n]$

Properties of LTI Systems

Invertibility for LTI systems

- Examples

Consider $h_0[n] = u[n]$, determine the inverse system $h_1[n]$

$$\because h_0[n] * h_1[n] = u[n] * h_1[n] \overset{hold}{=} \delta[n]$$

$$\delta[n] = u[n] - u[n-1] = u[n] * (\delta[n] - \delta[n-1])$$

$$\therefore h_1[n] = \delta[n] - \delta[n-1]$$

Properties of LTI Systems

Invertibility for LTI systems

- Examples

Consider the LTI system consisting of a pure time shift

$$y(t) = x(t - t_0),$$

determine the inverse system.

Properties of LTI Systems

Causality for LTI systems

- If $h[n] = 0$ for $\mathrm{n} < 0$, or $h(t) = 0$ for $t < 0$, the system is causal
- Equivalent to the condition of initial rest

$$y[n] = \sum_{k=-\infty}^{n} x[k]h[n-k] \quad \text{or} \quad y[n] = \sum_{k=0}^{\infty} h[k]x[n-k]$$

$$y(t) = \int_{-\infty}^{t} x(\tau)h(t-\tau)\,d\tau \quad \text{or} \quad y(t) = \int_{0}^{\infty} h(\tau)x(t-\tau)d\tau$$

Properties of LTI Systems

Causality for LTI systems

- Examples

  - Accumulator: $y[n] = \sum_{l=-\infty}^{n} x[l]$ Causal LTI system

$$h[n] = \sum_{l=-\infty}^{n} \delta[l] = u[n] \qquad h[n] = 0 \text{ for n} < 0$$

  - Factor 2 interpolator: $y[n] = x_u[n] + \frac{1}{2}(x_u[n-1] + x_u[n+1])$

Non-Causal LTI system

$$h[n] = \delta[n] + \frac{1}{2}(\delta[n-1] + \delta[n+1])$$

$$h[n] \neq 0 \text{ for n} = -1$$

Properties of LTI Systems

Stability for LTI systems

- A discrete LTI system is stable if $h[n]$ is absolutely summable

- A continuous LTI system is stable if $h(t)$ is absolutely integrable

$$\sum_{k=-\infty}^{\infty} |h[k]| < \infty$$ absolutely summable

$$\int_{-\infty}^{\infty} |h(\tau)| d\tau < \infty$$ absolutely integrable

Stability for LTI systems

- Proof: "if and only if" (Sufficient and necessary condition)

$$|y[n]| = \left|\sum_{k=-\infty}^{\infty} h[k]x[n-k]\right| \leq \sum_{k=-\infty}^{\infty} |h[k]x[n-k]| = \sum_{k=-\infty}^{\infty} |h[k]|\,|x[n-k]|$$

$$\therefore |y[n]| \leq \sum_{k=-\infty}^{\infty} |h[k]|\,|x[n-k]|$$

If $|x[n-k]| \leq B_x$ $\quad |y[n]| \leq B_x \sum_{k=-\infty}^{\infty} |h[k]|$

If and only if $\sum_{k=-\infty}^{\infty} |h[k]| < \infty$ $\quad |y[n]| < \infty$

Properties of LTI Systems

Stability for LTI systems

- Proof: continuous case

If $|x(t-\tau)| \le B_x$

$$|y(t)| = \left|\int_{-\infty}^{\infty} h(\tau)x(t-\tau)d\tau\right| \le \int_{-\infty}^{\infty} |h(\tau)| \cdot |x(t-\tau)|d\tau \le B_x \int_{-\infty}^{\infty} |h(\tau)|d\tau$$

If and only if $\int_{-\infty}^{\infty} |h(\tau)|d\tau < \infty$ $\quad$ $|y(t)| < \infty$

Properties of LTI Systems

Stability for LTI systems

- Examples

$$y[n] = x[n - n_0]$$

$$h[n] = \delta[n - n_0]$$

$$\sum_{n=-\infty}^{\infty} |h[n]| = \sum_{n=-\infty}^{\infty} |\delta[n - n_0]| = 1$$

Properties of LTI Systems

Stability for LTI systems

- Examples $h[n] = \alpha^n \mu[n]$

$$\sum_{n=-\infty}^{\infty} |h[n]| = \sum_{n=-\infty}^{\infty} |\alpha^n| \mu[n] = \sum_{n=0}^{\infty} |\alpha^n| = \frac{1}{1-|\alpha|} \quad \text{If } |\alpha| < 1$$

If $|\alpha| = 1$, the system is unstable

Properties of LTI Systems

The unit step response of LTI systems

- The unit step response, $s(t)$ or $s[n]$, corresponding to the output with input $x(t) = u(t)$ or $x[n] = u[n]$

$$s[n] = \mu[n] * h[n] = \sum_{k=-\infty}^{\infty} h[k]u[n-k] = \sum_{k=-\infty}^{n} h[k]$$

$$\mu[n] = \sum_{k=-\infty}^{n} \delta[k] \qquad\qquad s[n] = \sum_{k=-\infty}^{n} h[k]$$

$$h[n] = s[n] - s[n-1]$$

Properties of LTI Systems

The unit step response of LTI systems

- The unit step response, $s(t)$ or $s[n]$, corresponding to the output with input $x(t) = u(t)$ or $x[n] = u[n]$

$$s(t) = \mu(t) * h(t) = \int_{-\infty}^{\infty} h(\tau)u(t-\tau)d\tau = \int_{-\infty}^{t} h(\tau)d\tau$$

$$\mu(t) = \int_{-\infty}^{t} \delta(\tau)d\tau \qquad s(t) = \int_{-\infty}^{t} h(\tau)d\tau$$

$$h(t) = \frac{ds(t)}{dt} = s'(t)$$

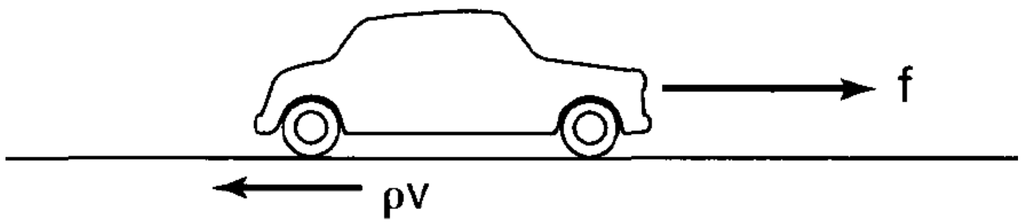
Linear Time-Invariant Systems (ch.2)

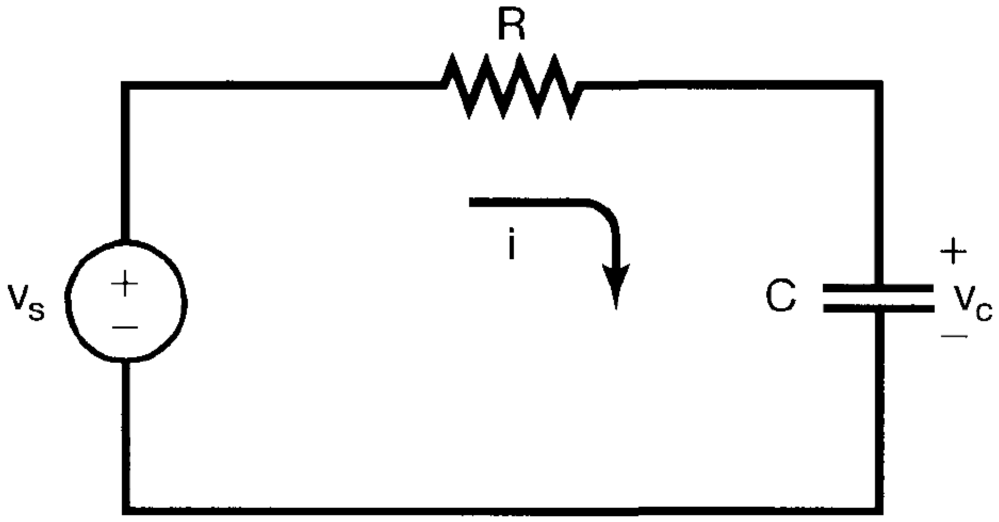
- ❑ Discrete-Time LTI Systems
- ❑ Continuous-Time LTI Systems
- ❑ Properties of LTI Systems
- ❑ Differential or Difference Equations

Differential or Difference Equations

Differential equation

- First order system





$$\frac{dv(t)}{dt} + \frac{\rho}{m}v(t) = \frac{1}{m}f(t)$$

$$\frac{dv_c(t)}{dt} + \frac{1}{RC}v_c(t) = \frac{1}{RC}v_s(t).$$

- In general: $$\frac{dy(t)}{dt} + ay(t) = bx(t)$$
- Describes a relationship between the input and the output (implicit)
- Auxiliary conditions are needed for solving the DE.

Differential or Difference Equations

Differential equation

- First order system: example

$$\frac{dy(t)}{dt} + 2y(t) = x(t)$$

If $x(t) = Ke^{3t}u(t)$ $\quad y(t) = ?$

- Solution:

$$y(t) = y_p(t) + y_h(t)$$

$y_p(t)$: particular solution, *forced response (same form as input)*

$y_h(t)$: Homogenous solution

$$\frac{dy(t)}{dt} + 2y(t) = 0$$

Differential equation

- First order system: example $$\frac{dy(t)}{dt} + 2y(t) = x(t)$$

If $x(t) = Ke^{3t}u(t)$ $\quad y(t) = ?$

- Particular solution: Let $y_p(t) = Ye^{3t}$, for t>0

$$3Ye^{3t} + 2Ye^{3t} = Ke^{3t} \longrightarrow Y = K/5 \longrightarrow y_p(t) = \frac{K}{5}e^{3t}$$

- Homogenous solution: Let $y_h(t) = Ae^{st}$, for t>0

$$Ase^{st} + 2Ae^{st} = 0 \longrightarrow s = -2 \longrightarrow y_h(t) = Ae^{-2t}$$

$$y(t) = Ae^{-2t} + \frac{K}{5}e^{3t}, \text{ for } t > 0$$

Differential or Difference Equations

Differential equation

$$y(t) = Ae^{-2t} + \frac{K}{5}e^{3t}, \text{ for } t > 0$$

- Auxiliary condition is required to determine *A*
- Initial rest as auxiliary condition for causal LTI systems: $y(0) = 0$

$$A + \frac{K}{5} = 0 \Rightarrow A = -\frac{K}{5} \Rightarrow y(t) = \frac{K}{5}(e^{3t}+e^{-2t}), \text{ for } t > 0$$

$$= \frac{K}{5}(e^{3t}+e^{-2t})u(t)$$

Differential or Difference Equations

Differential equation

- General case: Nth-order linear constant-coefficient differential equation

$$\sum_{k=0}^{N} a_k \frac{d^k y(t)}{dt^k} = \sum_{k=0}^{M} b_k \frac{d^k x(t)}{dt^k}$$

- Particular solution + Homogenous solution: $y(t) = y_p(t) + y_h(t)$
  - $y_p(t)$: *forced response (same form as input)*
  - $y_h(t)$: Natural response, $\sum_{k=0}^{N} a_k \frac{d^k y(t)}{dt^k} = 0$
- Initial rest as auxiliary condition, that is if $x(t) = 0$ for $t \leq t_0$,

$$y(t_0) = \frac{dy(t_0)}{dt} = \cdots = \frac{d^{N-1} y(t_0)}{dt^{N-1}} = 0$$

Differential or Difference Equations

Difference equation

- General case: *N*th-order linear constant-coefficient difference equation

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k]$$

- Particular solution + Homogenous solution: $y[n] = y_p[n] + y_h[n]$
  - $y_p[n]$: *forced response (same form as input)*
  - $y_h[n]$: Natural response, $\sum_{k=0}^{N} a_k y[n-k] = 0$
- Initial rest as auxiliary condition, that is if $x[n] = 0$ for $n \leq n_0$,

$$y[n_0] = y[n_0-1] = \cdots = y[n_0-(N-1)] = 0$$

Differential or Difference Equations

Difference equation

- Recursive solution:

$$y[n] = \frac{1}{a_0}\left\{\sum_{k=0}^{M} b_k x[n-k] - \sum_{k=1}^{N} a_k y[n-k]\right\}$$

- Particular case *N*=0

$$y[n] = \frac{1}{a_0}\sum_{k=0}^{M} b_k x[n-k]$$

Non-recursive equation

$$h[n] = \frac{1}{a_0}\sum_{k=0}^{M} b_k \delta[n-k]$$

Finite impulse response (FIR) system

Differential or Difference Equations

Difference equation

- Recursive solution: example $$y[n]-\frac{1}{2}y[n-1]=x[n]$$
  - Consider $x[n]=K\delta[n]$ and take initial rest: $y[-1]=0$

$$y[0]=x[0]+\frac{1}{2}y[-1]=K \qquad y[1]=x[1]+\frac{1}{2}y[0]=\frac{1}{2}K$$

$$y[2]=x[2]+\frac{1}{2}y[1]=\left(\frac{1}{2}\right)^2K \quad \dots \quad y[n]=x[n]+\frac{1}{2}y[n-1]=\left(\frac{1}{2}\right)^nK$$

$$\therefore h[n]=\left(\frac{1}{2}\right)^n u[n]$$

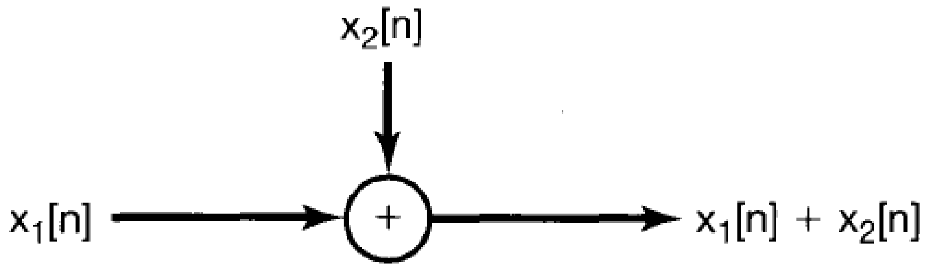
Infinite impulse response (IIR) system

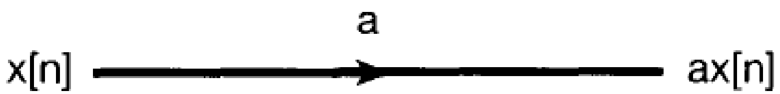
- Generally $\sum_{k=0}^{N}a_k y[n-k]=\sum_{k=0}^{M}b_k x[n-k]$ $\begin{cases}N=0, \text{FIR system}\\ N>0, \text{IIR system}\end{cases}$ Not always!

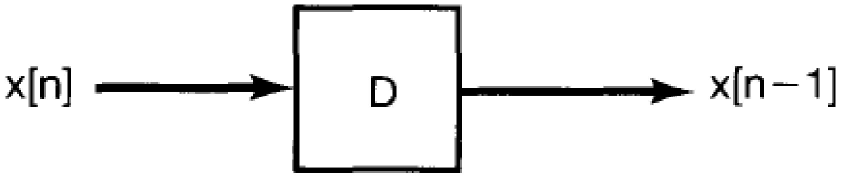
Differential or Difference Equations

Block Diagram Representations

- Basic elements: discrete-time

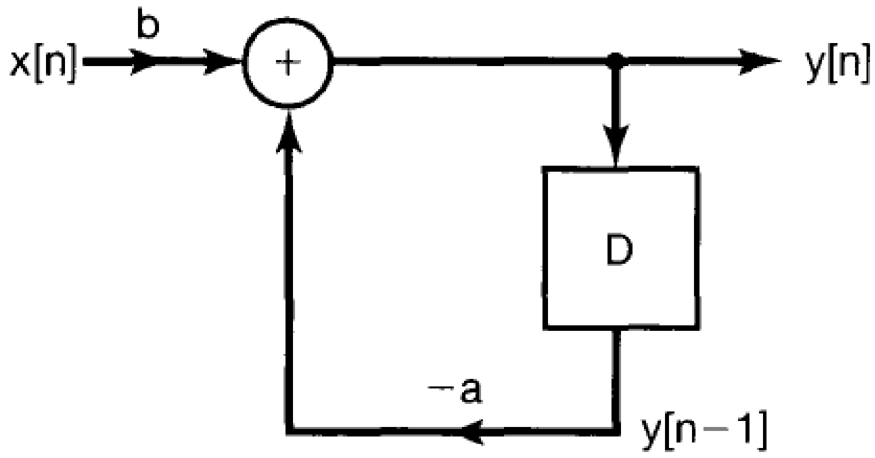






$$y[n] + ay[n-1] = bx[n]$$

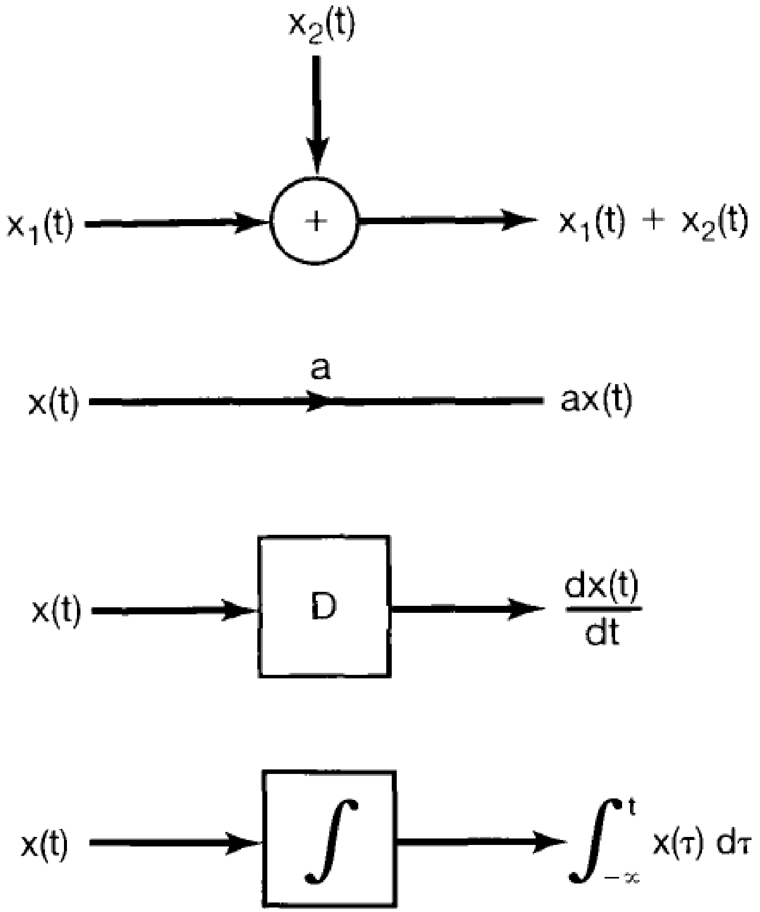
$$y[n] = -ay[n-1] + bx[n]$$



Differential or Difference Equations

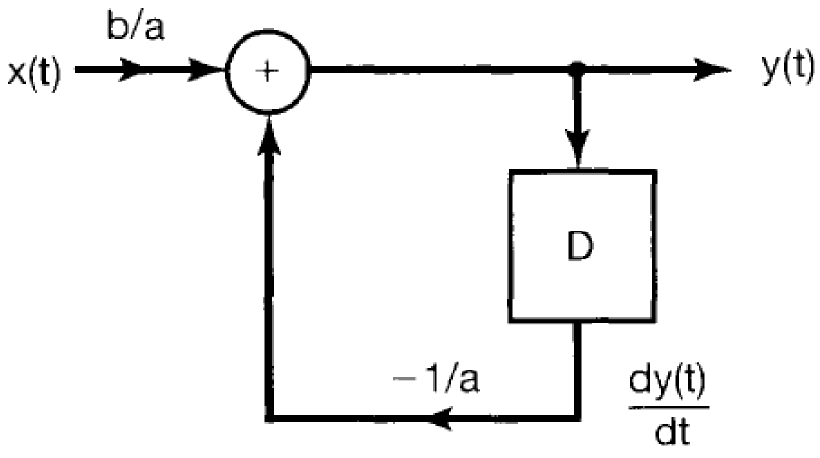
Block Diagram Representations

- Basic elements: continuous-time



$$\frac{dy(t)}{dt} + ay(t) = bx(t)$$

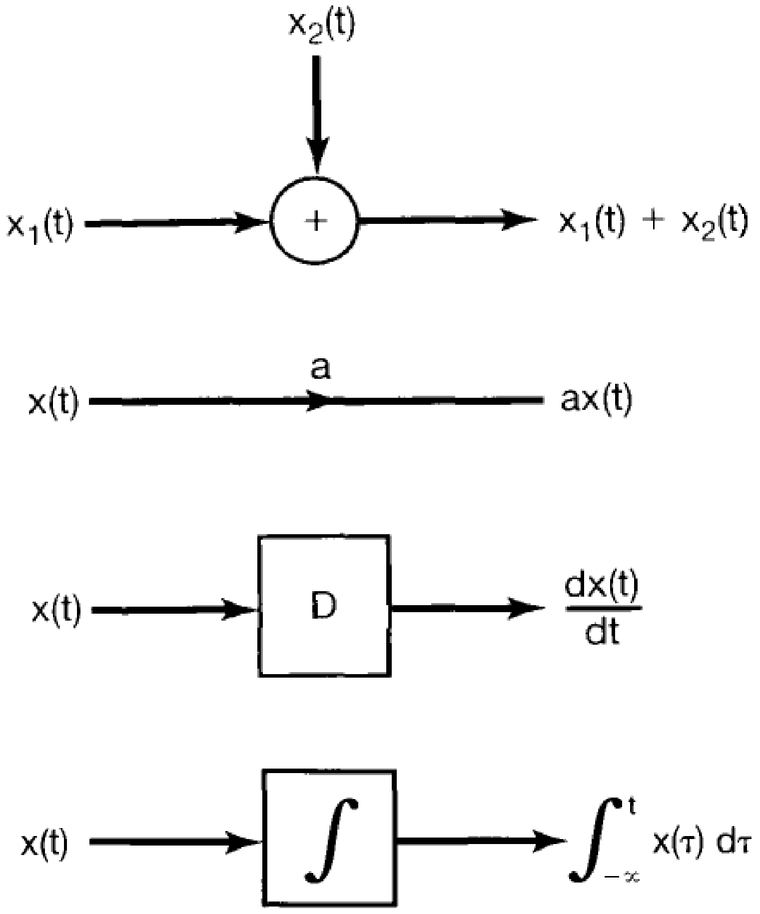
$$y(t) = -\frac{1}{a}\frac{dy(t)}{dt} + \frac{b}{a}x(t)$$



Differential or Difference Equations

Block Diagram Representations

- Basic elements: continuous-time



$$\frac{dy(t)}{dt} + ay(t) = bx(t)$$

$$\frac{dy(t)}{dt} = -ay(t) + bx(t)$$

$$y(t) = \int_{-\infty}^{t} [bx(\tau) - ay(\tau)]d\tau$$